Closure and Preferences

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Abstract

We investigate the results of Kreps (1979), dropping his completeness axiom. As an added generalization, we work on arbitrary lattices, rather than a lattice of sets. We show that one of the properties of Kreps is intimately tied with representation via a *closure operator*. That is, a preference satisfies Kreps' axiom (and a few other mild conditions) if and only if there is a closure operator on the lattice, such that preferences over elements of the lattice coincide with dominance of their closures. We tie the work to recent literature by Richter and Rubinstein (2015).

1 Introduction.

In behavioral decision theory, the term *menu* refers to a bundle of alternatives, any of which an individual may consume at a future date. The menu choice literature refers to individual choice amongst menus. The interpretation is that an individual chooses a menu, from which she will be asked to choose at some later date. Usually, this second stage of choice is only implicit; we never get to see this second stage. The menu choice literature is predicated on the observation that many individuals seek to "leave their options open," as they may be (informally) "uncertain" about what their future preferences over alternatives may be. For example, a typical preference discussed in the menu literature might feature choices such as:

$$\{\text{apple}, \text{banana}\} \succ \{\text{apple}\}$$

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and

$$\{\text{apple}, \text{banana}\} \succ \{\text{banana}\}.$$

Observe that a "rational" decision maker, who perfectly knows what her preferences will be at the later date when she will be expected to choose from the menu, would never exhibit such choices. She would either prefer the apple or the banana, and would therefore be indifferent between the menu consisting of both and *at least one* of the singleton menus.

The seminal work on menu choice is due to Kreps (1979). He establishes two classic decision-theoretic results on the theory of preferences over menus. First, he characterizes those preferences over menus that behave as what we will call *indirect preferences*. These are preferences for which there is an underlying preference over alternatives generating the preference over menus. Second, he characterizes those preferences over menus that admit a *preference for flexibility* representation.[1] Such a preference can be represented as if there is a collection of preferences over alternatives, and the preference over menus is monotonic with respect to these preferences.

Our aim in this note is to consider Kreps' first result without the completeness property, and establish that several interesting examples from the theory of choice share a common structure. We do so in a broader framework than menu choice—the key observation is that the set of "menus" has an algebraic structure (namely union and intersection) that renders it a semilattice. To this end, and observing that there is nothing particularly special about the collection of menus, we work on more general partially ordered sets satisfying different algebraic properties.

In the context of menu choice, our primary contribution here is to establish (in a suitably generalized environment) that upon dropping completeness (the fact that any pair of objects can be ranked), we admit a vector *indirect preference* representation. This means that there is a family of indirect preferences, and one menu dominates another if and only if it does so for every indirect preference in the family. This is termed "vector-valued," since if each indirect preference admits a real-valued representation, the ranking coincides with vector dominance of the imputed image of sets under the vector of representations. Such ideas are more or less standard in the theory of incomplete rankings (Szpilrajn, 1930; Ok, 2002). For example, dropping completeness from the remaining von Neumann-Morgenstern axioms admits a vector expected utility representation.

Much of the proof of this observation is already implicit in Kreps' work. In the proof of his second result (mentioned above), he defines an auxiliary binary relation on menus. This auxiliary relation can be demonstrated to have all of the properties of a relation in his first theorem, with the exception of completeness.

We actually go further, and work on more general semilattices. Our first result does the following, for a complete join semilattice. We study the analogues of Kreps' axioms in the first theorem, without completeness, and establish that there is a one-to-one correspon-

[1] A representation refers to a real-valued function whose ordinal structure coincides with the given choice behavior.

dence between orders satisfying these axioms and *closure operators* (Ward, 1942). Closure operators are objects from mathematics, but they have recently found much application in economics. For example, Richter and Rubinstein (2015) studies the family of *convex geometries*, which are a special type of closure. In another work, Nöldeke and Samuelson (2018) recently exploits the theory of Galois connections in mechanism design; Galois connections are intimately tied to the theory of closure.[2] There are many closure operators that are familiar in economics. Topological closure is a closure operator on the lattice of sets. The convex hull is a closure operator on a lattice of sets. The convex envelope of a real-valued function is a closure operator on the lattice of functions. Generally speaking, any object that can be defined as the "smallest" object of a certain type dominating another object serves as a closure. For example, the topological closure of a set is the smallest closed set containing that set. The convex hull is the smallest convex set containing that set, and so forth.

Kreps' work and most subsequent work focuses on the semilattice of menus. It might be thought that this is without loss of generality, as many semilattices or lattices are isomorphic to lattices of sets. In particular, the celebrated Birkhoff representation theorem (Birkhoff, 1937) claims that any distributive lattice can be homomorphically embedded in a lattice of sets with the usual union and intersection properties. Motivating our general exercise are two examples of rankings of *partitions* of a given set. The set of partitions is naturally ordered by the refinement relation, and is well-known to be non-distributive. In particular, these lattices cannot be mathematically modeled as lattices of sets. So the added generality of our results possesses implications for domains that need not "look like" lattices of sets.

On a lattice of subsets of a given set (with the usual union and intersection operations), it turns out that closure operators can be represented as the intersection of lower contour sets of weak orders.[3] This fact is implicit in Kreps and is entirely analogous to Richter and Rubinstein's observation that a convex geometry (antimatroid) can be represented as the intersection of lower contour sets of linear orders.[4] Closely related as well is the famous decomposition result for path-independent choice functions of Aizerman and Malishevski (1981). This allows the general "vector" representation alluded to for families of sets. See also Richter and Rubinstein (2019) for an abstract definition of convexity based on these ideas.

Now, we can also investigate Kreps' second result for an arbitrary lattice; positing the natural analogue of his axiom for an arbitrary lattice elucidates the structure of his second theorem. It allows us to define the same auxiliary relation defined in Kreps, which we are able to show satisfies the axioms of his first result. We can use these facts to establish a generalized version of the Kreps result: any preferences over an arbitrary lattice satisfying

[2] Every Galois connection induces a closure via the composition of the connection with its inverse; every closure can be induced from a Galois connection.

[3] A weak order is a binary relation that is complete and transitive. A lower contour set is the set of objects weakly dominated by some given object.

[4] Actually they show upper contour sets, but the idea is the same.

the adapted Kreps axioms can be represented as a strictly monotonic function of some closure operator. This is of practical relevance, as Chambers and Echenique (2008) and Chambers and Echenique (2009) jointly establish another result: preferences satisfying Kreps' axioms are those that have monotonic and submodular representations. Thus, it turns out that the ordinal content of submodularity is captured by the property of being strictly monotonic with respect to a closure operator.

Applications of these types of results can be found, for example, in Chambers and Miller (2014a,b, 2018). Here we detail a few.

Example 1. *Let Ω be a nonempty set, with $|\Omega| < +\infty$. Ω is intended to represent a set of* states of the world*, which may be known or unknown by a given decision maker.*

The set of partitions over Ω is denoted Π. The refinement *relation $\geq$ on Π is defined as:*

$$\pi \geq \pi'$$

if for every $A \in \pi$, there is $B \in \pi'$ for which $A \subseteq B$. In this case, we say that π refines *π'.*

The partitional framework is the canonical model for information about states of the world. Each element of a partition is an event*. If the decision maker possesses partition π and the true state is $\omega \in \Omega$, then the decision maker "knows" the member of π that contains ω. With this interpretation, refinement models the notion of "more informative than."*

Observe that given any two partitions, π and π', there is a coarsest common refinement, which we write $\pi \vee \pi' \in \Pi$. This object represents the conjunction of all information contained in the two partitions π and π'.

Now, imagine an intrinsic preference for information, as in Grant et al. (1998) or Masatlioglu et al. (2017). Consider an individual who possesses information $\pi^* \in \Pi$, which is (to an outside observer) unknown. This individual expresses preferences over information partitions, and it is reasonable to assume that she does so in accordance with the framework of Blackwell (1953). In particular, we imagine we can observe her preference over different information structures, for different menus of actions.

Now, this individual will evaluate information structure $\pi \in \Pi$ by joining it with her already possessed information π^*, resulting in information $(\pi^* \vee \pi)$. Thus, she will express a preference $\pi \succsim \pi'$ exactly when $(\pi \vee \pi^*) \geq (\pi' \vee \pi^*)$.

Now, suppose we want to test the hypothesis that this individual actually possesses an information structure π^*, which is unobservable to us. To this end, we wish to study the properties that $\succsim$ has. Let us address a (very) simple implication of the model here. Suppose that the individual expresses that $\pi \succsim \pi'$, and that she conforms to the model. We can infer then that $(\pi \vee \pi^*) \geq (\pi' \vee \pi^*)$. Consequently, $(\pi \vee \pi^*) \geq (\pi \vee \pi' \vee \pi^*)$, implying that $\pi \succsim (\pi \vee \pi')$.

In other words, if π is deemed "better" than π', it follows that joining the information π' to π can be of no benefit. This is not a general property of intrinsic preference for information; therefore, it can be understood as a basic testable implication of the model. As far as we know, this simple observation is new. And in particular, it provides a simple and direct method of testing when an individual may possess unobserved information.

Example 2. *The following example is mathematically very close to Example 1. Let* $N \equiv \{1, \ldots, n\}$*. A set* $I \subseteq N$ *is an* interval *if whenever* $x, y \in N$ *and* $x < y$*, if* $z \in N$ *for which* $x < z < y$*, then* $z \in I$*.*[5] *Observe that for any partition* $\pi \in \Pi$*, there is a unique coarsest partition* $F(\pi) \in \Pi$ *for which each* $A \in F(\pi)$ *is an interval, and whenever* $A \in F(\pi)$*, then there is* $B \in \pi$ *for which* $A \subseteq B$ *(the coarsest refinement).*

Imagine now that the elements of N *are individuals, and the ordering reflects (somehow) their spatial location. Perhaps the individuals are partitioned into "types," so that an element of a partition* π *reflects individuals of the same type, without respect to location. A geographic planner who wants to build a road is charged with not breaking up communities of individuals consisting of the same type. Perhaps this is done in order to preserve a neighborhood structure. To this end, the geographic planner prefers as refined a partition as possible. Now, there may be a set* $A \in \pi$ *that is not an interval, in which case that type is already broken up. In such a situation, the geographic planner does not need to worry about preserving the neighborhood structure; all that is relevant is to preserve neighboring types. To that end, the planner prefers* π *to* π' *exactly when* $F(\pi)$ *refines* $F(\pi')$*;* i.e. $\pi \succsim \pi'$ *iff* $F(\pi)$ *refines* $F(\pi')$*.*

Observe now that this structure also satisfies some a very restrictive property: if $\pi \succsim \pi'$*, then the coarsest common refinement* $(\pi \vee \pi')$ *does not add any advantage to the geographic planner: that is* $(\pi \vee \pi') \sim \pi$*.*

While it may be easy enough to check explicitly whether the geographic planner has in mind this particular model when the ordering of N *is given, if that ordering is* not *given (for example, in the case of a non-geographic example), this simple axiom provides a simple test as to whether there actually is such an order. Again, we believe this result is novel.*

2 The model.

Let $(X, \leq)$ be a *partially ordered set*, *i.e.*, a set endowed with a binary relation that satisfies the following properties:

Reflexivity: For all $x \in X$, $x \leq x$.

Antisymmetry: For all $x, y \in X$, $x \leq y$ and $y \leq x$ imply $x = y$.

Transitivity: For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

The least upper bound of two (or more) elements $x, y \in X$ according to $\leq$ is referred to as their *join* and is denoted $x \vee y$; the greatest lower bound of these elements is referred to as their *meet* and is denoted $x \wedge y$. The partially ordered set $(X, \leq)$ is a *join semilattice* if $x \vee y$ exists for all $x, y \in X$, and is *join complete* if $\bigvee x_i$ exists for every subset $\{x_i\} \subseteq X$. The partially ordered set $(X, \leq)$ is a *meet semilattice* if $x \wedge y$ exists for all $x, y \in X$, and is *meet complete* if $\bigwedge x_i$ exists for every subset $\{x_i\} \subseteq X$. A partially ordered set is called a

[5] Here, $<$ is the usual "less than" relation on the natural numbers, of which N is a subset.

lattice if it is both a join and meet semilattice; it is furthermore called a *complete lattice* if it is meet and join complete. A subset $\mathcal{C} \subseteq X$ of a partially ordered set is called a *chain* if for all $x, y \in \mathcal{C}$, either $x \leq y$ or $y \leq x$.

A *closure operator* is a function $c : X \to X$ that satisfies the following three properties:

Extensivity: For all $x \in X$, $x \leq c(x)$.

Monotonicity: For all $x, y \in X$, $x \leq y$ implies $c(x) \leq c(y)$.

Idempotence: For all $x \in X$, $c(c(x)) = c(x)$.

We are interested in investigating binary relations $\succsim$ on X. In addition to reflexivity, antisymmetry, and transitivity that we have defined above; we are also interested in the following properties:

Join dominance: For all $x, y \in X$, if $x \succsim y$, then $x \succsim (x \vee y)$.

Meet dominance: For all $x, y \in X$, if $x \succsim y$, then $(x \wedge y) \succsim y$.

Monotonicity: For all $x, y \in X$, if $x \geq y$, then $x \succsim y$.

Completeness: For all $x, y \in X$, $x \succsim y$ or $y \succsim x$.

Lower semicontinuity: For any chain $\mathcal{C}$ and $x \in X$,
if for all $y \in \mathcal{C}$, $x \succsim y$, then $x \succsim \bigvee_{y \in \mathcal{C}} y$.

Note that join dominance of $\succsim$ is defined when $(X, \leq)$ is a join semilattice and lower semicontinuity of $\succsim$ is defined when $(X, \leq)$ is join complete. Likewise, meet dominance of $\succsim$ is defined when $(X, \leq)$ is a meet semilattice.

Observe that, when all chains of $(X, \leq)$ are finite, lower semicontinuity of $\succsim$ is vacuous, as $\bigvee_{y \in \mathcal{C}} y \in \mathcal{C}$.

Lower semicontinuity is, in our view, a technical property, in the sense that no empirical test utilizing only finite data could refute it. On the other hand, the other properties are inherently testable. Our point here is not to point out a deep mathematical contribution, but rather to isolate some properties common to disparate methods of choice on lattices. Our focus is on empirically meaningful statements, so we have isolated continuity from the remaining properties.

One could, of course, hypothesize as primitive assumptions certain conjunctions of these axioms. Continuity is formally a property of lower semicontinuity of the binary relation with respect to the order topology generated by $\leq$.

In addition, observe that monotonicity of $\succsim$ implies reflexivity of $\succsim$.

Lemma 1. *Suppose that $(X, \leq)$ is join semilattice. If a binary relation $\succsim$ satisfies transitivity, monotonicity, and join dominance, then the following holds for all $x, y, z \in X$:*

$$x \succsim y \text{ and } x \succsim z \text{ imply } x \succsim (y \vee z).$$

Proof. Let $x \succsim y$ and $x \succsim z$. Since $x \succsim y$, we get $x \succsim (x \vee y)$ by join dominance. Furthermore, $(x \vee y) \geq x$ implies $(x \vee y) \succsim x$ by monotonicity. By transitivity, $(x \vee y) \succsim z$. Thus, by join dominance, $(x \vee y) \succsim (x \vee y \vee z)$. By transitivity and monotonicity, we get $x \succsim (x \vee y \vee z) \succsim (y \vee z)$. □

Theorem 1. *Suppose that $(X, \leq)$ is join complete. A binary relation $\succsim$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity if and only if there is a closure operator c on X for which $x \succsim y$ if and only if $c(x) \geq c(y)$. Suppose, furthermore, that $(X, \leq)$ is meet semilattice. Then, the binary relation $\succsim$ further satisfies meet dominance if and only if c further satisfies $c(x \wedge y) = c(x) \wedge c(y)$ for all $x, y \in X$.*

Proof. Suppose there is a closure c for which $c(x) \geq c(y)$ if and only if $x \succsim y$. Monotonicity is satisfied: if $x \geq y$ then $c(x) \geq c(y)$ by monotonicity of the closure, which implies $x \succsim y$. Join dominance is satisfied: suppose $x \succsim y$. Then $c(x) \geq c(y)$ by the hypothesis. By extensitivity of the closure, $c(x) \geq x$ and $c(y) \geq y$, so $(c(x) \vee c(y)) \geq (x \vee y)$. Therefore, idempotence and monotonicity of the closure imply that $c(x) = c(c(x)) = c(c(x) \vee c(y)) \geq c(x \vee y)$, where the second equality follows from $c(x) \geq c(y)$. Thus, $x \succsim (x \vee y)$ by the hypothesis. Finally, lower semicontinuity is satisfied: suppose $\mathcal{C}$ is a chain such that for all $y \in \mathcal{C}$, $x \succsim y$. Then by the hypothesis $c(x) \geq c(y)$ and by extensitivity of closure $c(y) \geq y$, so $c(x) \geq y$ for all $y \in \mathcal{C}$. Hence $c(x) \geq \bigvee_{y \in \mathcal{C}} y$ since $(X, \leq)$ is join complete. Using idempotence and monotonicity of c we get $c(x) = c(c(x)) \geq c(\bigvee_{y \in \mathcal{C}} y)$, or $x \succsim \bigvee_{y \in \mathcal{C}} y$.

Suppose further that the meet homomorphism property is satisfied: $c(x \wedge y) = c(x) \wedge c(y)$. Let $x \succsim y$. By the hypothesis $c(x) \geq c(y)$, which implies $c(x) \wedge c(y) = c(y)$. Therefore, by meet homomorphism, we get $c(x \wedge y) = c(y)$. Therefore, $(x \wedge y) \succsim y$ by the hypothesis, so meet dominance is satisfied.

Conversely, suppose that a binary relation $\succsim$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity. Define, for every $x \in X$,

$$c(x) = \bigvee \{z : x \succsim z\}.$$

We use a standard transfinite induction argument to prove that $c(x) \sim x$.[6]

Let us well-order the set $\{y : x \succsim y\}$, and call the resulting ordinal Λ, with order $\leq^*$. Thus, we write $\{y : x \succsim y\}$ as $\{y_\lambda\}_{\lambda \in \Lambda}$. Let us define $z_\lambda = \bigvee_{\lambda' \leq^* \lambda} y_{\lambda'}$.

We claim that $x \succsim z_\lambda$ for all $\lambda \in \Lambda$. There are three cases to consider.

1. First, in case $\lambda = 0$, the result is obvious as $z_0 = y_0 \in \{y : x \succsim y\}$.

2. In case λ is a successor ordinal, we know that $x \succsim z_{\lambda-1}$ and $x \succsim y_\lambda$. By Lemma 1, $x \succsim (z_{\lambda-1} \vee y_\lambda) = z_\lambda$.

[6] Here, $\sim$ denotes the indifference relation associated with $\succsim$.

3. In the third case, λ is a limit ordinal. Now, observe that $\{z_{\lambda'} : \lambda' < \lambda\}$ can be identified with an $\leq^*$-chain (by identifying any pair λ', λ'' for which $z_{\lambda'} = z_{\lambda''}$). Therefore, since $x \succsim z_{\lambda'}$ for each $\lambda' <^* \lambda$, we can apply lower semicontinuity and get $x \succsim \bigvee_{\lambda':\lambda'<^*\lambda} z_{\lambda'}$. Now, since $x \succsim y_\lambda$, we can apply Lemma 1 to get $x \succsim ((\bigvee_{\lambda':\lambda'<^*\lambda} z_{\lambda'}) \vee y_\lambda) = \bigvee_{\lambda':\lambda'\leq^*\lambda} y_{\lambda'} = z_{\lambda'}$.

Hence, we have shown that $x \succsim z_\lambda$ for all $\lambda \in \Lambda$. Now, observe that $\{z_\lambda : \lambda \in \Lambda\}$ can also be identified with an $\leq^*$-chain (by identifying any pair λ, λ' for which $z_\lambda = z_{\lambda'}$). And further observe that $\bigvee_{\lambda\in\Lambda} z_\lambda = \bigvee_{\lambda\in\Lambda} y_\lambda = c(x)$. Hence, by lower semicontinuity, we get that $x \succsim c(x)$. But since $x \in \{y : x \succsim y\} = \{y_\lambda\}_{\lambda\in\Lambda}$, $c(x) = \bigvee_{\lambda\in\Lambda} y_\lambda \geq x$, so monotonicity implies $c(x) \succsim x$. Therefore, $x \sim c(x)$. Now, we show that the properties of c defining a closure are satisfied.

1. Extensivity: $x \in \{z : x \succsim z\}$, so $c(x) = \bigvee\{z : x \succsim z\} \geq x$.

2. Monotonicity: Suppose $x \geq y$. By monotonicity of $\succsim$, we obtain $x \succsim y$. Then, if $y \succsim z$, it follows by transitivity that $x \succsim z$, so $\{z : x \succsim z\} \supseteq \{z : y \succsim z\}$. Therefore, $c(x) = \vee\{z : x \succsim z\} \geq \vee\{z : y \succsim z\} = c(y)$.

3. Idempotence: Since $x \sim c(x)$, the sets $\{z : x \succsim z\}$ and $\{z : c(x) \succsim z\}$ coincide. Therefore, $c(x) = \vee\{z : x \succsim z\} = \vee\{z : c(x) \succsim z\} = c(c(x))$.

Next, we show that $c(x) \geq c(y)$ if and only if $x \succsim y$. Suppose that $x \succsim y$. This implies that, as in the proof of monotonicity, $c(x) \geq c(y)$. Conversely, if $c(x) \geq c(y)$, monotonicity of the relation $\succsim$ implies that $c(x) \succsim c(y)$. Since $c(x) \sim x$ and $c(y) \sim y$, we conclude that $x \succsim y$.

Finally, suppose that $(X, \leq)$ is a meet semilattice and that the relation $\succsim$ satisfies meet dominance. Let $x, y \in X$. Monotonicity of the relation $\succsim$ implies that $c(x) \succsim (c(x) \wedge c(y))$ and $c(y) \succsim (c(x) \wedge c(y))$. Since $c(x) \sim x$ and $c(y) \sim y$, we get that $x \succsim (c(x) \wedge c(y))$ and $y \succsim (c(x) \wedge c(y))$. By meet dominance, $(c(x) \wedge c(y) \wedge x) \succsim (c(x) \wedge c(y))$. By monotonicity and transitivity, $y \succsim (c(x) \wedge c(y) \wedge x)$. Thus, meet dominance implies $(c(x) \wedge c(y) \wedge x \wedge y) \succsim (c(x) \wedge c(y) \wedge x)$. Conclude by transitivity that $(c(x) \wedge c(y) \wedge x \wedge y) \succsim (c(x) \wedge c(y))$. Again using the fact that for arbitrary a, $c(a) = \bigvee\{z : a \succsim z\}$, we conclude that $(c(x) \wedge c(y)) \leq c(c(x) \wedge c(y) \wedge x \wedge y)$. But $c(x) \wedge c(y) \wedge x \wedge y = x \wedge y$ since c is extensive; conclude $(c(x) \wedge c(y)) \leq c(x \wedge y)$. Finally, we know that $c(x) \geq c(x \wedge y)$ and $c(y) \geq c(x \wedge y)$ (by monotonicity of the closure c) so that $c(x) \wedge c(y) \geq c(x \wedge y)$, implying $c(x \wedge y) = c(x) \wedge c(y)$. $\square$

Let us now consider a property of a binary relation R on X, which is anticipated by several papers. See, for example, (Kreps, 1979; Nehring, 1999; Epstein and Marinacci, 2007; Chambers and Echenique, 2008).

Ordinal submodularity: For all $x, y, z \in X$, if $x \; R \; (x \vee y)$, then $(x \vee z) \; R \; (x \vee y \vee z)$.

It is easy to see that ordinal submodularity is implied by the combination of transitivity, monotonicity, and join dominance: if $x\ R\ (x \vee y)$, then by monotonicity $(x \vee z)\ R\ x$, so by transitivity $(x \vee z)\ R\ (x \vee y)$, and hence $(x \vee z)\ R\ (x \vee y \vee z)$ by join dominance.

The content of the following result is that even more than Theorem 1 is true: by weakening join dominance to ordinal submodularity, we can describe relations that are monotonic with respect to the ranking given by *some* closure operator. However, the relation need not *coincide* with the ranking given by the closure.

In the following result, we use the notation R for the primitive binary relation reflecting preference. This is because we will endogenously derive another relation $\succsim$ in the proof, which will satisfy the hypotheses of Theorem 1.

Theorem 2. *Suppose R is transitive and lower semicontinuous. Then R is monotonic and ordinally submodular if and only if there is a closure operator c on X for which the following are satisfied:*

1. *$c(x) \geq c(y)$ implies $x\ R\ y$, and*
2. *$c(x) > c(y)$ implies $x\ P\ y$.*[7]

Again, when all chains are finite, lower semicontinuity is vacuous.

Remark 1. *Note that hypothesis 2 is crucial here. An axiomatization of relations for which there exists a closure so that property 1 alone is satisfied would be based on completeness, transitivity, and monotonicity alone. To see this, consider the identity closure $c(x) = x$ and observe that 1 is satisfied by monotonicity. Hence, the additional content of ordinal submodularity is in adding the* strict *monotonicity hypothesis. In Kreps (1979), this is reflected by all of the states being non-null. In Dekel et al. (2001), the corresponding theorem allows null states. Presumably, by adding ordinal submodularity there, we would recover non-nullity. This feature of ordinal submodularity is also what allows Epstein and Marinacci (2007) to derive their result.*

Proof. Suppose that R is monotonic and ordinally submodular. We proceed to show that 1 and 2 are satisfied.

Define a new relation $\succsim$ by $x \succsim y$ if $x\ R\ (x \vee y)$. We claim that $\succsim$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity.

Transitivity follows thusly: Suppose $x \succsim y$ and $y \succsim z$. Then $x\ R\ (x \vee y)$ and $y\ R\ (y \vee z)$. By ordinal submodularity applied to the last statement, we have $(x \vee y)\ R\ (x \vee y \vee z)$. By transitivity, $x\ R\ (x \vee y \vee z)$ and by monotonicity, $(x \vee y \vee z)\ R\ (x \vee z)$, so that again by transitivity, $x\ R\ (x \vee z)$; conclude $x \succsim z$.

Monotonicity also follows: suppose that $x \geq y$. Since $(x \vee y) = x$, so $x\ R\ (x \vee y)$; hence $x \succsim y$.

[7] Here, P is the strict part of R.

Join dominance follows as if $x \succsim y$, then $x\ R\ (x \vee y)$. Now, $x \succsim (x \vee y)$ if $x\ R\ (x \vee (x \vee y)) = (x \vee y)$, which holds.

Lower semicontinuity of $\succsim$ follows from lower semicontinuity of R: let $\mathcal{C}$ be a chain such that for all $y \in \mathcal{C}$, $x \succsim y$. Then for all $y \in \mathcal{C}$, $x\ R\ (x \vee y)$. Note then that the collection $\{x \vee y : y \in \mathcal{C}\}$ can also be identified with a chain, so that $x\ R\ \left(\bigvee_{y \in \mathcal{C}} (x \vee y)\right)$. This implies $x\ R\ (x \bigvee_{y \in \mathcal{C}} y)$, which is equivalent to $x \succsim \bigvee_{y \in \mathcal{C}} y$.

Hence, Theorem 1 implies that there is a closure c such that $x \succsim y$ if and only if $c(x) \geq c(y)$.

Now, suppose $c(x) \geq c(y)$. We claim that $x\ R\ y$. Since $c(x) \geq c(y)$, we know that $x \succsim y$, from which we conclude $x\ R\ (x \vee y)$ and since $(x \vee y)\ R\ y$ by monotonicity, we get by transitivity of R that $x\ R\ y$.

Suppose further that $c(y) \geq c(x)$ is false. We claim that $x\ P\ y$. Suppose by means of contradiction that $y\ R\ x$. Then we have $x\ I\ y$,[8] and since $x\ R\ (x \vee y)$, we get by transitivity of R that $y\ R\ (x \vee y)$, so that $y \succsim x$, implying $c(y) \geq c(x)$, a contradiction.

Now, we show the converse that if there is such a closure then R satisfies monotonicity and ordinal submodularity.

To see that monotonicity of relation R is satisfied, suppose $x \geq y$. Then $c(x) \geq c(y)$ by monotonicity of the closure c and hence $x\ R\ y$, so monotonicity is satisfied.

To see that ordinal submodularity is satisfied, let $x, y, z \in X$ and suppose that $x\ R\ (x \vee y)$. Since $(x \vee y) \geq x$, it follows that $c(x \vee y) \geq c(x)$ by monotonicity of the closure c. If in fact $c(x \vee y) > c(x)$, we would have $(x \vee y)\ P\ x$, which is false. Thus $c(x \vee y) = c(x)$. Monotonicity of the closure c implies that $c(x \vee z) \geq c(x)$ and $c(x \vee z) \geq c(z)$, so $c(x \vee z) \geq (c(x) \vee c(z)) = (c(x \vee y) \vee c(z)) \geq (x \vee y \vee z)$ where the last inequality follows from extensitivity. Hence by monotonicity of c, $c(c(x \vee z)) \geq c(x \vee y \vee z)$. By idempotence of c, we have $c(x \vee z) \geq c(x \vee y \vee z)$. Hence $(x \vee z)\ R\ (x \vee y \vee z)$. □

Some of the ideas in this proof appear in Kreps (1979).

Let us now show that a closure operator on a join-semilattice of nonempty subsets of some given set can be written as an intersection of weak upper contour sets of some family of weak orders (where a weak order is a relation $\succsim$ that is complete and transitive).

Theorem 3. *Let $(X, \leq)$ be a join-complete semilattice of nonempty subsets of some set Y, with the usual intersection and union operations. Then $c : X \to X$ is a closure operator if and only if there is a family $\mathcal{W}$ of weak orders on Y for which $c(A) = \bigcap_{\succsim \in \mathcal{W}} \{x : \exists y \in A, y \succsim x\}$.*

This result can be compared to Richter and Rubinstein (2015), which establishes a related result for linear orders. Theorem 3 is implicit in Kreps. Theorems 1 and 3, taken together, imply Theorem 3 of Chambers and Miller (2018).

Proof. Showing that if there is a family $\mathcal{W}$ generating c in this fashion, then c is a closure is simple and left to the reader.

[8] I denotes the indifference relation corresponding to R.

For the converse, we construct $\mathcal{W}$ as follows. For each fixed point of c, where a fixed point is a set for which $c(A) = A$, we define a relation represented by the following utility function:

$$u_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Let $A \in X$ and let $x \in c(A)$. We want to show that for all $\succsim \in \mathcal{W}$, there is $y \in A$ for which $y \succsim x$. So, fix any $c(B) = B$. If $u_B(x) = 0$, then clearly there is $y \in A$ for which $y \succsim x$. On the other hand, suppose that $u_B(x) = 1$. This means that $x \notin B$. We need to show that there is $y \in A$ where $y \notin B$ as well. If, in fact $y \in A$ implies $y \in B$, then we would have $c(A) \subseteq c(B)$ by monotonicity of the closure, so that $c(A) \subseteq B$, contradicting $x \notin B$.

Conversely, suppose that for all $\succsim \in \mathcal{W}$, there is $y \in A$ for which $y \succsim x$. In particular, consider the relation induced by $u_{c(A)}$. Since $y \in A \subseteq c(A)$, $u_{c(A)}(y) = 0$, so we must have $u_{c(A)}(x) = 0$, implying $x \in c(A)$.

This construction is due to Kreps; ultimately, many other constructions would work (for example, one could construct the set of weak orders according to chains of fixed points; this is what Kreps does, but he only considers binary chains, which is enough). □

One could ask whether there is a similar result when requiring the family $\mathcal{W}$ to consist of orders satisfying reasonable *economic* properties. To this end, let $\leq^*$ be a partial order on some underlying set Y; we will be interested in nonempty subsets of Y.[9] Suppose now that each element $B \in 2^Y$ under consideration is *comprehensive*, in the sense that if $x \in B$ and $y \leq^* x$, then $y \in B$. We say that a binary relation $\succsim$ is *monotone* with respect to $\leq^*$ if whenever $x \leq^* y$, we have $y \succsim x$.

Theorem 4. *Let $\leq^*$ be a partial order on Y, and let $(X, \leq)$ be a lattice of nonempty and comprehensive subsets of 2^Y, with the usual intersection and union operations. Then $c : X \to X$ is a closure operator if and only if there is a family $\mathcal{W}$ of weak orders on Y that are monotone with respect to $\leq^*$ for which $c(A) = \bigcap_{\succsim \in \mathcal{W}} \{x : \exists y \in A$ such that $y \succsim x\}$.*

Proof. Observe that the construction in Theorem 3 results in weak orders that are monotone in the case in which each $B \in X$ is comprehensive. □

3 Conclusion

The purpose of this technical note is twofold: to establish a version of Kreps' (1979) classic result without completeness and to show that much of the analysis in that paper extends to more general lattices. Central to our analysis is the concept of a closure operator. It remains to characterize environments for which there is either more structure on the closure operator, or in the ranking of sets, when the weak orders can be endowed with more structure. For

[9] A partial order is reflexive, transitive, and antisymmetric.

example, while Theorem 4 gives conditions for weakly monotone orders, one could ask for strictly monotone orders. Likewise, in a suitably convex environment, one could ask for convex weak orders.

Other questions relate to comparative statics across closure operators. On a finite lattice, the set of closure operators is also a lattice. A useful exercise is to understand how these closures interact.

Finally we mention a closely related strand of literature: path independent choice functions, as defined in Plott (1973), are those which can be defined as the "extreme points" of certain closure operators on the lattice of sets (again the convex geometries). See Koshevoy (1999); Johnson and Dean (2001); Danilov and Koshevoy (2005) for example. Whether one can define interesting generalizations of these concepts for general closure operators is unknown to us. Chambers and Yenmez (2017) utilizes these ideas in the context of matching.

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