Closure and Preferences

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Abstract

We investigate the results of Kreps (1979), dropping his completeness axiom. As an added generalization, we work on arbitrary lattices, rather than a lattice of sets. We show that one of the properties of Kreps is intimately tied with representation via a *closure operator*. That is, a preference satisfies Kreps' axiom (and a few other mild conditions) if and only if there is a closure operator on the lattice, such that preferences over elements of the lattice coincide with dominance of their closures. We tie the work to recent literature by Richter and Rubinstein (2015). Finally, we carry the concept to the theory of path-independent choice functions.

1 Introduction

Kreps (1979) establishes two classic decision-theoretic results on the theory of preferences over menus. First, he characterizes those preferences over menus which behave as what we will call *indirect preferences*. These are preferences for which there is an underlying preference over alternatives generating the preference over menus. Second, he characterizes those preferences over menus which admit a *preference for flexibility* representation. Such a preference can be represented as if there is a collection of preferences over alternatives, and the preference over menus is monotonic with respect to these preferences.

Our aim in this note is to consider Kreps' first result without the completeness property. Interestingly, we establish that upon dropping completeness, we admit a vector *indirect preference* representation. Such ideas are more or less standard in the theory of incomplete preferences (Szpilrajn, 1930; Ok, 2002). For example, dropping completeness from the remaining von Neumann-Morgenstern axioms admits a vector expected utility representation.

In fact, much of the proof of this result is implicit in Kreps' work itself. In the proof of his second result, he defines an auxiliary relation. This auxiliary relation can be demonstrated to have all of the properties of a relation in his first theorem, with the exception of completeness.

In so doing we elucidate the structure of Kreps' second result. While Kreps works on the lattice of finite sets, there is nothing particularly special about this lattice.

Our first result does the following, for an arbitrary finite lattice (this can be generalized somewhat). We study the analogues of Kreps' axioms in the first theorem, without completeness, and establish that there is a one-to-one correspondence between orders satisfying these axioms and *closure operators* (Ward, 1942). Closure operators are objects from mathematics, but they have recently found much application in economics. For example, Richter and Rubinstein (2015) study the family of *convex geometries*, which are a special type of closure. In another work, Nöldeke and Samuelson (2015) recently exploit the theory of Galois connections in mechanism design; Galois connections are intimately tied to the theory of closure.[1] There are many closure operators which are familiar in economics. Topological closure is a closure operator on the lattice of sets. The convex hull is a closure operator on a lattice of sets. The convex envelope of a real-valued function is a closure operator on the lattice of functions. Generally speaking, any object which can be defined as the "smallest" object of a certain type dominating another object serves as a closure. For example, the topological closure of a set is the smallest closed set containing that set. The convex hull is the smallest convex set containing that set, and so forth.

On a lattice of subsets of a given set (with the usual union and intersection operations), it turns out that closure operators can be represented as the intersection of lower contour sets of weak orders. This fact is also implicit in Kreps and is entirely analogous to Richter and Rubinstein's observation that a convex geometry (antimatroid) can be represented as the intersection of lower contour sets of linear orders.[2] Closely related as well is the famous decomposition result for path-independent choice functions of Aizerman and Malishevski (1981). This allows the general "vector" representation alluded to for families of sets.

Now, we can also investigate Kreps' second result for an arbitrary lattice; positing the natural analogue of his axiom for an arbitrary lattice elucidates the structure of his second theorem. It allows us to define the same auxiliary relation defined in Kreps; which we are able to show satisfies the axioms of his first result. We can use these facts to establish a generalized version of the Kreps result: any preferences over an arbitrary lattice satisfying the adapted Kreps axioms can be represented as a strictly monotonic function of some closure operator. This is especially interesting, as Chambers and Echenique (2008) and Chambers and Echenique (2009) jointly establish another result: preferences satisfying Kreps' axioms are those which have monotonic and submodular representations. Thus, it turns out that the ordinal content of submodularity is captured by the property of being strictly monotonic with respect to a closure operator.

Applications of these types of results can be found, for example, in Chambers and Miller (2014a,b, 2015).

2 The model

Let $(X, \leq)$ be a *partially ordered set*, *i.e.*, a set endowed with a binary relation that satisfies the following properties:

- (*reflexivity*) for all $x \in X$, $x \leq x$;

[1]Every Galois connection induces a closure via the composition of the connection with its inverse; every closure can be induced from a Galois connection.

[2]Actually they show upper contour sets, but the idea is the same.

- (*antisymmetry*) for all $x, y \in X$, $x \leq y$ and $y \leq x$ imply $x = y$; and
- (*transitivity*) for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

The least upper bound of two (or more) elements $x, y \in X$ according to $\leq$ is referred to as their *join* and is denoted $x \vee y$; the greatest lower bound of these elements is referred to as their *meet* and is denoted $x \wedge y$. The partially ordered set $(X, \leq)$ is a *join semilattice* if $x \vee y$ exists for all $x, y \in X$, and is *join complete* if $\bigvee x_i$ exists for every subset $\{x_i\} \in X$. The partially ordered set $(X, \leq)$ is a *meet semilattice* if $x \wedge y$ exists for all $x, y \in X$, and is *meet complete* if $\bigwedge x_i$ exists for every subset $\{x_i\} \in X$. A partially ordered set is called a *lattice* if it is both a join and meet semilattice; it is furthermore called a *complete lattice* if it is meet and join complete. A subset $\mathcal{C} \subseteq X$ of a partially ordered set is called a chain if for all $x, y \in \mathcal{C}$, either $x < y$ or $y < x$.

A *closure operator* is a function $c : X \to X$ that satisfies the following three properties:

- (*extensivity*) for all $x \in X$, $x \leq c(x)$;
- (*monotonicity*) for all $x, y \in X$, $x \leq y$ implies $c(x) \leq c(y)$; and
- (*idempotence*) for all $x \in X$, $c(c(x)) = c(x)$.

We are interested in investigating relations $\succeq$ on X. In addition to reflexivity, antisymmetry, and transitivity that we have defined above; we are also interested in the following properties:

- (*join dominance*) for all $x, y \in X$, if $x \succeq y$, then $x \succeq (x \vee y)$;
- (*meet dominance*) for all $x, y \in X$, if $x \succeq y$, then $(x \wedge y) \succeq y$;
- (*monotonicity*) for all $x, y \in X$, if $x \geq y$, then $x \succeq y$;
- (*completeness*) for all $x, y \in X$, $x \succeq y$ or $y \succeq x$;
- (*lower semicontinuity*) for any chain $\mathcal{C}$ and $x \in X$, if for all $y \in \mathcal{C}$, $x \succeq y$, then $x \succeq \bigvee_{y \in \mathcal{C}} y$; and
- (*upper semicontinuity*) for any chain $\mathcal{C}$ and $x \in X$, if for all $y \in \mathcal{C}$, $y \succeq x$, then $\bigwedge_{y \in \mathcal{C}} y \succeq x$.

Note that join dominance and lower semicontinuity are defined when $(X, \leq)$ is join complete. Likewise, meet dominance and upper semicontinuity are defined when $(X, \leq)$ is meet complete. Observe that, when all chains are finite, lower semicontinuity is vacuous, as $\bigvee_{y \in \mathcal{C}} y \in \mathcal{C}$. Similarly, upper semicontinuity is vacuous when all chains are finite because $\bigwedge_{y \in \mathcal{C}} y \in \mathcal{C}$. In addition, observe that monotonicity implies reflexivity.

Lemma 1. *Suppose that $(X, \leq)$ is join complete. If a binary relation $\succeq$ satisfies transitivity, monotonicity, and join dominance, then the following holds for all $x, y, z \in X$:*

$$x \succeq y \text{ and } x \succeq z \text{ imply } x \succeq (y \vee z).$$

Proof. Let $x \succeq y$ and $x \succeq z$. Since $x \succeq y$, we get $x \succeq (x \vee y)$ by join dominance. Furthermore, $(x \vee y) \geq x$ implies $(x \vee y) \succeq x$ by monotonicity. By transitivity, $(x \vee y) \succeq z$. Thus, by join dominance, $(x \vee y) \succeq (x \vee y \vee z)$. By transitivity and monotonicity, we get $x \succeq (x \vee y \vee z) \succeq (y \vee z)$. □

Theorem 1. *Suppose that $(X, \leq)$ is join complete. A binary relation $\succeq$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity if and only if there is a closure operator c on X for which $x \succeq y$ if and only if $c(x) \geq c(y)$. Suppose, furthermore, that $(X, \leq)$ is meet complete. Then, the binary relation $\succeq$ further satisfies meet dominance if and only if c further satisfies $c(x \wedge y) = c(x) \wedge c(y)$ for all $x, y \in X$.*

Proof. Suppose there is a closure c for which $c(x) \geq c(y)$ if and only if $x \succeq y$. Monotonicity is satisfied: if $x \geq y$ then $c(x) \geq c(y)$ by monotonicity of the closure which implies $x \succeq y$. Join dominance is satisfied: suppose $x \succeq y$. Then $c(x) \geq c(y)$ by the hypothesis. By extensitivity of the closure, $c(x) \geq x$ and $c(y) \geq y$, so $(c(x) \vee c(y)) \geq (x \vee y)$. Therefore, idempotence and monotonicity of the closure imply that $c(x) = c(c(x)) = c(c(x) \vee c(y)) \geq c(x \vee y)$, where the second equality follows from $c(x) \geq c(y)$. Thus, $x \succeq (x \vee y)$ by the hypothesis. Finally, lower semicontinuity is satisfied: suppose $\mathcal{C}$ is a chain such that for all $y \in \mathcal{C}$, $x \succeq y$. Then by the hypothesis $c(x) \geq c(y)$ and by extensitivity of closure $c(y) \geq y$, so $c(x) \geq y$ for all $y \in \mathcal{C}$. Hence $c(x) \geq \bigvee_{y \in \mathcal{C}} y$ since $(X, \leq)$ is join complete. Using idempotence and monotonicity of c we get $c(x) = c(c(x)) \geq c(\bigvee_{y \in \mathcal{C}} y)$, or $x \succeq \bigvee_{y \in \mathcal{C}} y$.

Suppose further that the meet homomorphism property is satisfied: $c(x \wedge y) = c(x) \wedge c(y)$. Let $x \succeq y$. By the hypothesis $c(x) \geq c(y)$, which implies $c(x) \wedge c(y) = c(y)$. Therefore, by meet homomorphism, we get $c(x \wedge y) = c(y)$. Therefore, $(x \wedge y) \succeq y$ by the hypothesis, so meet dominance is satisfied.

Conversely, suppose that a binary relation $\succeq$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity. Define, for every $x \in X$,

$$c(x) = \bigvee \{z : x \succeq z\}.$$

We use transfinite induction to prove that $c(x) \sim x$. Let us well-order the set $\{y : x \succeq y\}$, and call the resulting ordinal Λ, with order $\leq^*$. Thus, we write $\{y : x \succeq y\}$ as $\{y_\lambda\}_{\lambda \in \Lambda}$. Let us define $z_\lambda = \bigvee_{\lambda' \leq^* \lambda} y_{\lambda'}$.

We claim that $x \succeq z_\lambda$ for all $\lambda \in \Lambda$. There are three cases to consider.

1. First, in case $\lambda = 0$, the result is obvious as $z_0 = y_0 \in \{y : x \succeq y\}$.

2. In case λ is a successor ordinal, we know that $x \succeq z_{\lambda-1}$ and $x \succeq y_\lambda$. By Lemma 1, $x \succeq (z_{\lambda-1} \vee y_\lambda) = z_\lambda$.

3. In the third case, λ is a limit ordinal. Now, observe that $\{z_{\lambda'} : \lambda' < \lambda\}$ can be identified with an $\leq^*$-chain (by identifying any pair λ', λ'' for which $z_{\lambda'} = z_{\lambda''}$). Therefore, since $x \succeq z_{\lambda'}$ for each $\lambda' <^* \lambda$, we can apply lower semicontinuity and get $x \succeq \bigvee_{\lambda' : \lambda' <^* \lambda} z_{\lambda'}$. Now, since $x \succeq y_\lambda$, we can apply Lemma 1 to get $x \succeq ((\bigvee_{\lambda' : \lambda' <^* \lambda} z_{\lambda'}) \vee y_\lambda) = \bigvee_{\lambda' : \lambda' \leq^* \lambda} y_{\lambda'} = z_{\lambda'}$.

Hence, we have shown that $x \succeq z_\lambda$ for all $\lambda \in \Lambda$. Now, observe that $\{z_\lambda : \lambda \in \Lambda\}$ can also be identified with an $\leq^*$-chain (by identifying any pair λ, λ' for which $z_\lambda = z_{\lambda'}$). And further observe that $\bigvee_{\lambda \in \Lambda} z_\lambda = \bigvee_{\lambda \in \Lambda} y_\lambda = c(x)$. Hence, by lower semicontinuity, we get that $x \succeq c(x)$. But since $x \in \{y : x \succeq y\} = \{y_\lambda\}_{\lambda \in \Lambda}$, $c(x) = \bigvee_{\lambda \in \Lambda} y_\lambda \geq x$, so monotonicity implies $c(x) \succeq x$. Therefore, $x \sim c(x)$.[3]

Now, we show that the properties of c defining a closure are satisfied.

- Extensivity: $x \in \{z : x \succeq z\}$, so $c(x) = \bigvee\{z : x \succeq z\} \geq x$.
- Monotonicity: Suppose $x \geq y$. By monotonicity of $\succeq$, we obtain $x \succeq y$. Then, if $y \succeq z$, it follows by transitivity that $x \succeq z$, so $\{z : x \succeq z\} \supseteq \{z : y \succeq z\}$. Therefore, $c(x) = \vee\{z : x \succeq z\} \geq \vee\{z : y \succeq z\} = c(y)$.
- Idempotence: Since $x \sim c(x)$, the sets $\{z : x \succeq z\}$ and $\{z : c(x) \succeq z\}$ coincide. Therefore, $c(x) = \vee\{z : x \succeq z\} = \vee\{z : c(x) \succeq z\} = c(c(x))$.

Next, we show that $c(x) \geq c(y)$ if and only if $x \succeq y$. Suppose that $x \succeq y$. This implies that, as in the proof of monotonicity, $c(x) \geq c(y)$. Conversely, if $c(x) \geq c(y)$, monotonicity of the relation $\succeq$ implies that $c(x) \succeq c(y)$. Since $c(x) \sim x$ and $c(y) \sim y$, we conclude that $x \succeq y$.

Finally, suppose that $(X, \leq)$ is meet complete and the relation $\succeq$ satisfies meet dominance. Let $x, y \in X$. Monotonicity of the relation $\succeq$ implies that $c(x) \succeq (c(x) \wedge c(y))$ and $c(y) \succeq (c(x) \wedge c(y))$. Since $c(x) \sim x$ and $c(y) \sim y$, we get that $x \succeq (c(x) \wedge c(y))$ and $y \succeq (c(x) \wedge c(y))$. By meet dominance, $(c(x) \wedge c(y) \wedge x) \succeq (c(x) \wedge c(y))$. By monotonicity and transitivity, $y \succeq (c(x) \wedge c(y) \wedge x)$. Thus, meet dominance implies $(c(x) \wedge c(y) \wedge x \wedge y) \succeq (c(x) \wedge c(y) \wedge x)$. Conclude by transitivity that $(c(x) \wedge c(y) \wedge x \wedge y) \succeq (c(x) \wedge c(y))$. Again using the fact that for arbitrary a, $c(a) = \bigvee\{z : a \succeq z\}$, we conclude that $(c(x) \wedge c(y)) \leq c(c(x) \wedge c(y) \wedge x \wedge y)$. But $c(x) \wedge c(y) \wedge x \wedge y = x \wedge y$ since c is extensive; conclude $(c(x) \wedge c(y)) \leq c(x \wedge y)$. Finally, we know that $c(x) \geq c(x \wedge y)$ and $c(y) \geq c(x \wedge y)$ (by monotonicity of the closure c) so that $c(x) \wedge c(y) \geq c(x \wedge y)$, implying $c(x \wedge y) = c(x) \wedge c(y)$. □

Let us now consider a new property of a relation R on X:

- (*ordinal submodularity*) for all $x, y, z \in X$, if $x \; R \; (x \vee y)$, then $(x \vee z) \; R \; (x \vee y \vee z)$.

It is easy to see that ordinal submodularity is implied by the combination of transitivity, monotonicity, and join dominance: if $x \; R \; (x \vee y)$, then by monotonicity $(x \vee z) \; R \; x$, so by transitivity $(x \vee z) \; R \; (x \vee y)$, and hence $(x \vee z) \; R \; (x \vee y \vee z)$ by join dominance.

The content of the following result is that even more than Theorem 1 is true: by weakening join dominance to ordinal submodularity, we can describe relations which are monotonic with respect to the ranking given by a closure operator. However, the relation need not *coincide* with the ranking given by the closure. The result relies on the axiom of choice (as do any further results requiring lower semicontinuity, without further mention).

[3]Here, $\sim$ denotes the indifference relation identified with $\succeq$.

Theorem 2. *Suppose R is transitive and lower semicontinuous. Then R is monotonic and ordinally submodular if and only if there is a closure operator c on X for which the following are satisfied:*

1. *$c(x) \geq c(y)$ implies $x \; R \; y$, and*
2. *$c(x) > c(y)$ implies $x \; P \; y$.*[4]

Again, when all chains are finite, lower semicontinuity is vacuous.

Remark 1. *Note that hypothesis 2 is crucial here. An axiomatization of relations for which there exists a closure so that property 1 alone is satisfied would be based on completeness, transitivity, and monotonicity alone. To see this, consider the identity closure $c(x) = x$ and observe that 1 is satisfied by monotonicity. Hence, the additional content of ordinal submodularity is in adding* strict *monotonicity hypothesis. In Kreps (1979), this is reflected by all of the states being non-null. In Dekel et al. (2001), the corresponding theorem allows null states. Presumably, by adding ordinal submodularity there, we would recover non-nullity. This feature of ordinal submodularity is also what allows Epstein and Marinacci (2007) to derive their result.*

Proof. Suppose that R is monotonic and ordinally submodular. We proceed to show that 1 and 2 are satisfied.

Define a new relation $\succeq$ by $x \succeq y$ if $x \; R \; (x \vee y)$. We claim that $\succeq$ satisfies transitivity, monotonicity, join dominance, and lower semicontinuity.

Transitivity follows thusly: Suppose $x \succeq y$ and $y \succeq z$. Then $x \; R \; (x \vee y)$ and $y \; R \; (y \vee z)$. By ordinal submodularity applied to the last statement, we have $(x \vee y) \; R \; (x \vee y \vee z)$. By transitivity, $x \; R \; (x \vee y \vee z)$ and by monotonicity, $(x \vee y \vee z) \; R \; (x \vee z)$, so that again by transitivity, $x \; R \; (x \vee z)$; conclude $x \succeq z$.

Monotonicity also follows: suppose that $x \geq y$. Since $(x \vee y) = x$, so obviously $x \; R \; (x \vee y)$; hence $x \succeq y$.

Join dominance follows as if $x \succeq y$, then $x \; R \; (x \vee y)$. Now, $x \succeq (x \vee y)$ if $x \; R \; (x \vee (x \vee y)) = (x \vee y)$, which holds.

Lower semicontinuity of $\succeq$ follows from lower semicontinuity of R: let $\mathcal{C}$ be a chain such that for all $y \in \mathcal{C}$, $x \succeq y$. Then for all $y \in \mathcal{C}$, $x \; R \; (x \vee y)$. Note then that the collection $\{x \vee y : y \in \mathcal{C}\}$ can also be identified with a chain, so that $x \; R \; \left(\bigvee_{y \in \mathcal{C}} (x \vee y)\right)$. This implies $x \; R \; (x \bigvee_{y \in \mathcal{C}} y)$ which is equivalent to $x \succeq \bigvee_{y \in \mathcal{C}} y$.

Hence, Theorem 1 implies that there is a closure c such that $x \succeq y$ if and only if $c(x) \geq c(y)$.

Now, suppose $c(x) \geq c(y)$. We claim that $x \; R \; y$. Since $c(x) \geq c(y)$, we know that $x \succeq y$, from which we conclude $x \; R \; (x \vee y)$ and since $(x \vee y) \; R \; y$ by monotonicity, we get by transitivity of R that $x \; R \; y$.

Suppose further that $c(y) \geq c(x)$ is false. We claim that $x \; P \; y$. Suppose by means of contradiction that $y \; R \; x$. Then we have $x \; I \; y$,[5] and since $x \; R \; (x \vee y)$, we get by transitivity of R that $y \; R \; (x \vee y)$, so that $y \succeq x$, implying $c(y) \geq c(x)$, a contradiction.

[4] Here, P is the strict part of R.

[5] I denotes the indifference relation corresponding to R.

Now, we show the converse that if there is such a closure then R satisfies monotonicity and ordinal submodularity.

To see that monotonicity of relation R is satisfied, suppose $x \geq y$. Then $c(x) \geq c(y)$ by monotonicity of the closure c and hence $x \; R \; y$, so monotonicity is satisfied.

To see that ordinal submodularity is satisfied, let $x, y, z \in X$ and suppose that $x \; R \; (x \vee y)$. Since $(x \vee y) \geq x$, it follows that $c(x \vee y) \geq c(x)$ by monotonicity of the closure c. If in fact $c(x \vee y) > c(x)$, we would have $(x \vee y) \; P \; x$, which is false. Thus $c(x \vee y) = c(x)$. Monotonicity of the closure c implies that $c(x \vee z) \geq c(x)$ and $c(x \vee z) \geq c(z)$, so $c(x \vee z) \geq (c(x) \vee c(z)) = (c(x \vee y) \vee c(z)) \geq (x \vee y \vee z)$ where the last inequality follows from extensitivity. Hence by monotonicity of c, $c(c(x \vee z)) \geq c(x \vee y \vee z)$. By idempotence of c, we have $c(x \vee z) \geq c(x \vee y \vee z)$. Hence $(x \vee z) \; R \; (x \vee y \vee z)$. □

Some of the ideas in this proof appear in Kreps (1979).

Let us now show that a closure operator on a lattice of nonempty subsets of some given set can be written as an intersection of weak upper contour sets of some family of weak orders (where a weak order is a relation $\succeq$ which is complete and transitive).

Theorem 3. *Let $(X, \leq)$ be a lattice of nonempty subsets of some set X, with the usual intersection and union operations. Then $c : X \to X$ is a closure operator if and only if there is a family $\mathcal{W}$ of weak orders for which $c(A) = \bigcap_{\succeq \in \mathcal{W}} \{x : \exists y \in A \text{ such that } y \succeq x\}$.*

This result can be compared to Richter and Rubinstein (2015), which establishes a related result for linear orders. Theorem 3 is implicit in Kreps.

Proof. Showing that if there is a family $\mathcal{W}$ generating c in this fashion, then c is a closure is simple and left to the reader.

For the converse, we construct $\mathcal{W}$ as follows. For each fixed point of c, where a fixed point is a set for which $c(A) = A$, we define a relation represented by the following utility function:

$$u_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Let A be an arbitrary set and let $x \in c(A)$. We want to show that for all $\succeq \in \mathcal{W}$, there is $y \in A$ for which $y \succeq x$. So, fix any $c(B) = B$. If $u_B(x) = 0$, then clearly there is $y \in A$ for which $y \succeq x$. On the other hand, suppose that $u_B(x) = 1$. This means that $x \notin B$. We need to show that there is $y \in A$ where $y \notin B$ as well. If, in fact $y \in A$ implies $y \in B$, then we would have $c(A) \subseteq c(B)$ by monotonicity of the closure, so that $c(A) \subseteq B$, contradicting $x \notin B$.

Conversely, suppose that for all $\succeq \in \mathcal{W}$, there is $y \in A$ for which $y \succeq x$. In particular, consider the relation induced by $u_{c(A)}$. Since $y \in A \subseteq c(A)$, $u_{c(A)}(y) = 0$, so we must have $u_{c(A)}(x) = 0$, implying $x \in c(A)$.

This construction is due to Kreps; ultimately, many other constructions would work (for example, one could construct the set of weak orders according to chains of fixed points; this is what Kreps does, but he only considers binary chains, which is enough). □

One could ask whether there is a similar result when requiring the family $\mathcal{W}$ to consist of orders satisfying reasonable *economic* properties. To this end, let $\leq^*$ be a partial order.[6] Suppose now that each element of X is *comprehensive*, in the sense that if $x \in X$ and $y \leq^* x$, then $y \in X$. We say that a binary relation $\succeq$ is *monotone* with respect to $\leq^*$ if whenever $x \leq^* y$, we have $y \succeq x$.

Theorem 4. *Let $(X, \leq)$ be a lattice of nonempty and comprehensive subsets of some set X, with the usual intersection and union operations. Then $c : X \to X$ is a closure operator if and only if there is a family $\mathcal{W}$ of weak orders which are monotone with respect to $\leq^*$ for which $c(A) = \bigcap_{\succeq \in \mathcal{W}} \{x : \exists y \in A$ such that $y \succeq x\}$.*

Proof. Observe that the construction in Theorem 3 results in weak orders which are monotone in the case in which each X is comprehensive. □

A more difficult result would ask when such a family could also be taken to be *strictly monotonic* with respect to some $<^* \subseteq \leq^*$. Roughly, this happens if the set of fixed points of c is "rich" enough. But some technical assumptions are necessary. Suppose that each element of X is *finitely generated*, in the sense that for each $A \in X$, there is $\{a_1, \ldots, a_K\}$ for which $A = \bigcup_k \{y : y \leq^* a_k\}$. Then one can define an order on X by $A \ll B$ if for all $a_k \in A$, there is $b_l \in B$ for which $a_k <^* b_l$. In such a case, the extension required of Theorem 4 that would ensure that each $\succeq \in \mathcal{W}$ is strictly monotone would be that for any fixed point A of c, if $A \ll B$, then there is a fixed point D for which $A \subsetneq D \subseteq B$. We leave the details to the reader.[7]

2.1 Path Independence

A choice correspondence $C : P(X) \rightrightarrows P(X)$ is such that for every $Y \subseteq X$ and $Z \in C(Y)$ we have $Z \subseteq Y$. For a set of subsets $\mathcal{Y} \subseteq P(X)$, we define $C(\mathcal{Y}) \equiv \bigcup_{Y \in \mathcal{Y}} C(Y)$. Note that this definition implies that for two sets of subsets $\mathcal{Y}, \mathcal{Z} \subseteq P(X)$, $\mathcal{Y} \subseteq \mathcal{Z}$ implies that $C(\mathcal{Y}) \subseteq C(\mathcal{Z})$.

For any two sets $\mathcal{Y}, \mathcal{Z} \subseteq P(X)$, let $\mathcal{Y} \uplus \mathcal{Z} \equiv \{W : \exists Y \in \mathcal{Y}$ and $\exists Z \in \mathcal{Z}$ such that $W = Y \cup Z\}$.

A choice correspondence C is **path independent** if for every $Y, Z \subseteq X$,

$$C(Y \cup Z) = C(C(Y) \uplus C(Z)).$$

Plott (1973) defined path-independent choice functions. The definition above is a generalization to choice correspondences.

For a path independent C and any $Y \subseteq X$, define $c\ell : P(X) \to P(X)$ as

$$c\ell(Y) = \bigcup_{Z : C(Y) \subseteq C(Z)} Z.$$

First, we show the following basic property of $c\ell$.

[6] A partial order is reflexive, transitive, and antisymmetric.

[7] The construction in the previous two results would not work. Instead, one would need to construct preferences in $\mathcal{W}$ as being generated by maximal chains of fixed points of c.

Lemma 2. *For all $Y \subseteq X$, $C(Y) \subseteq C(c\ell(Y))$.*

Proof. Let $Z_1, \ldots, Z_k \subseteq X$ be such that $C(Y) \subseteq C(Z_i)$ for every i and $\cup Z_i = c\ell(Y)$. By construction, $C(Y) \subseteq C(Z_1) \uplus \ldots \uplus C(Z_k)$; consequently, because C is monotonic when applied to sets of subsets, $C(C(Y)) \subseteq C(C(Z_1) \uplus \ldots \uplus C(Z_k))$. By path independence, $C(C(Z_1) \uplus \ldots \uplus C(Z_k)) = C(Z_1 \cup \ldots \cup Z_k) = C(c\ell(Y))$, so $C(C(Y)) \subseteq C(c\ell(Y))$. Path independence implies that $C(C(Y)) = C(Y)$. Therefore, we conclude that $C(Y) \subseteq C(c\ell(Y))$. $\square$

Next, we show that $c\ell$ is a closure operator on $P(X)$.

Lemma 3. *The function $c\ell$ is a closure operator.*

Proof. Extensivity is trivial.

To show idempotence, note that $c\ell(Y) \subseteq c\ell(c\ell(Y))$ by extensitivity. By Lemma 2 $C(Y) \subseteq C(c\ell(Y))$. Thus for every Z such that $C(c\ell(Y)) \subseteq C(Z)$, we have $C(Y) \subseteq C(Z)$. By the definition of $c\ell$ this implies that $c\ell(c\ell(Y)) \subseteq c\ell(Y)$. Therefore, $c\ell$ is idempotent.

To show monotonicity, suppose that $Y \subseteq Z$. By Lemma 2, $C(Y) \subseteq C(c\ell(Y))$ and $C(Z) \subseteq C(c\ell(Z))$, consequently $C(Y) \uplus C(Z) \subseteq C(c\ell(Y)) \uplus C(c\ell(Z))$. Because C is monotone with respect to sets of subsets, it follows that $C(C(Y) \uplus C(Z)) \subseteq C(C(c\ell(Y)) \uplus C(c\ell(Z)))$. By path independence, it follows that $C(Z) = C(Y \cup Z) \subseteq C(c\ell(Y) \cup c\ell(Z))$. By the definition of $c\ell$ it follows that $c\ell(Y) \cup c\ell(Z) \subseteq c\ell(Z)$ and therefore that $c\ell(Y) \subseteq c\ell(Z)$. $\square$

Using this closure operator, we establish the lattice structure of path-independent choice correspondences.

Theorem 5. *Consider a path-independent choice correspondence C. Define the following binary relation $\succeq$ on the choice of closed sets, $\{C(c\ell(Y)) : Y \subseteq X\}$:*

$$C(c\ell(Y)) \succeq C(c\ell(Z)) \Longleftrightarrow c\ell(Y) \supseteq c\ell(Z).$$

Then $\succeq$ defines a lattice where the meet and join operators are defined as

- $C(c\ell(Y)) \vee C(c\ell(Z)) = C(c\ell(Y \cup Z))$, *and*
- $C(c\ell(Y)) \wedge C(c\ell(Z)) = C(c\ell(Y) \cap c\ell(Z))$.

Proof. It is easy to see that $\succeq$ is a partial order.

Let us show $C(c\ell(Y \cup Z))$ is the join of $C(c\ell(Y))$ and $C(c\ell(Z))$. First note that $C(c\ell(Y \cup Z)) \succeq C(c\ell(Y))$ and $C(c\ell(Y \cup Z)) \succeq C(c\ell(Z))$ because $c\ell(Y \cup Z) \supseteq c\ell(Y)$ and $c\ell(Y \cup Z) \supseteq c\ell(Z)$ which follows from the monotonicity of $c\ell$. Hence, $C(c\ell(Y \cup Z))$ is an upper bound of $C(c\ell(Y))$ and $C(c\ell(Z))$. Let $C(c\ell(W))$ be another upper bound. By definition, $c\ell(W) \supseteq c\ell(Y)$ and $c\ell(W) \supseteq c\ell(Z)$, so $c\ell(W) \supseteq c\ell(Y) \cup c\ell(Z) \supseteq Y \cup Z$ as $c\ell$ satisfies extensivity. Because of monotonicity, applying $c\ell$ to both sides yields, $c\ell(c\ell(W)) \supseteq c\ell(Y \cup Z)$. This is equivalent to $c\ell(W) \supseteq c\ell(Y \cup Z)$ because of idempotence of $c\ell$. Therefore, we get $C(c\ell(W)) \succeq C(c\ell(Y \cup Z))$ by definition of $\succeq$. Hence, $C(c\ell(Y \cup Z))$ is the join (or least upper bound) of $C(c\ell(Y))$ and $C(c\ell(Z))$.

Now we prove that $C(cl(cl(Y) \cap cl(Z)))$ is the meet of $C(cl(Y))$ and $C(cl(Z))$. Since $cl(Y) \supseteq cl(Y) \cap cl(Z)$, we get $cl(cl(Y)) \supseteq cl(cl(Y) \cap cl(Z))$ by monotonicity of cl. Since cl is idempotent, this reduces to $cl(Y) \supseteq cl(cl(Y) \cap cl(Z))$. Hence, $C(cl(Y)) \succeq C(cl(cl(Y) \cap cl(Z)))$ by definition of $\succeq$. Analogously, we get that $C(cl(Z)) \succeq C(cl(cl(Y) \cap cl(Z)))$. Therefore, $C(cl(cl(Y) \cap cl(Z)))$ is a lower bound of $C(cl(Y))$ and $C(cl(Z))$. Let $C(cl(W))$ be another lower bound: $C(cl(Y)) \succeq C(cl(W))$ and $C(cl(Z)) \succeq C(cl(W))$. By definition of $\succeq$, $cl(Y) \supseteq cl(W)$ and $cl(Z) \supseteq cl(W)$. As a result, $cl(Y) \cap cl(Z) \supseteq cl(W)$. Applying cl to both sides yields $cl(cl(Y) \cap cl(Z)) \supseteq cl(cl(W))$ by monotonicity. Idempotence implies $cl(cl(Y) \cap cl(Z)) \supseteq cl(W)$. Therefore, we get $C(cl(cl(Y) \cap cl(Z))) \succeq C(cl(W))$ by definition of $\succeq$. Hence, $C(cl(cl(Y) \cap cl(Z)))$ is the meet (or greatest lower bound) of $C(cl(Y))$ and $C(cl(Z))$.

Finally, we prove that $cl(cl(Y) \cap cl(Z)) = cl(Y) \cap cl(Z)$. By monotonicity and idempotence of the closure, $cl(cl(Y) \cap cl(Z)) \subseteq cl(cl(Y)) = cl(Y)$ and $cl(cl(Y) \cap cl(Z)) \subseteq cl(cl(Z)) = cl(Z)$. As a result, $cl(cl(Y) \cap cl(Z)) \subseteq cl(Y) \cap cl(Z)$. But extensitivity implies $cl(Y) \cap cl(Z) \subseteq cl(cl(Y) \cap cl(Z))$. As a result, $cl(cl(Y) \cap cl(Z)) = cl(Y) \cap cl(Z)$. □

This generalizes the result that path-independent choice functions have a lattice structure (Johnson and Dean, 2001).

Next we investigate the Aizerman-Malishevski decomposition of path-independent choice correspondences.

Lemma 4. *$Z \in C(Y)$ implies $cl(Y) = cl(Z)$.*

Proof. Suppose that $Z \in C(Y)$. Since $C(C(Y)) = C(Y)$ by path independence, we have that $C(Z) \subseteq C(Y)$. This implies $cl(Y) \subseteq cl(Z)$ by the construction of cl. By monotonicity of cl, we also have that $cl(Y) \supseteq cl(Z)$ since $Y \supseteq Z$. Therefore, $cl(Y) = cl(Z)$. □

A choice correspondence C is **non-nested** if for every $Y \subseteq X$ and two distinct $Z, W \in C(Y)$, $Z \not\subseteq W$ and $W \not\subseteq Z$.

Lemma 5. *Consider a path-independent and non-nested choice correspondence C. If $Y \subseteq Z \subseteq cl(Y)$ and $Y \in C(cl(Y))$, then $Y \in C(Z)$.*

Proof. By path independence $C(cl(Y)) = C((cl(Y) \setminus Z) \cup Z) = C(C(cl(Y) \setminus Z) \uplus C(Z))$. Since $Y \in C(cl(Y))$ and $Y \subseteq Z$, there must exist $Z' \in C(Z)$ such that $Y \subseteq Z'$.

By Lemma 2, $C(Z) \subseteq C(cl(Z))$. From monotonicity and idempotence of cl it follows that $cl(Y) = cl(Z)$ (since $Y \subseteq Z \subseteq cl(Y)$), and therefore $C(Z) \subseteq C(cl(Y))$ and $Z' \in C(cl(Y))$. Since $Y, Z' \in C(cl(Y))$, $Y \subseteq Z'$, and C is non-nested, it follows that $Y = Z'$. Hence, $Y \in C(Z)$. □

Theorem 6. *Consider a path-independent and non-nested choice correspondence C. Then $C(Y) = \min\{Z \subseteq Y : cl(Z) = cl(Y)\}$.*

Proof. We first show that for $W \in \min\{Z \subseteq Y : cl(Z) = cl(Y)\}$, we have $W \in C(cl(W))$. To see this, let $W' \in C(W)$. By Lemma 4, $cl(W') = cl(W)$. Then $W' = W$ by the construction of W and so $C(W) = \{W\}$. By Lemma 2, $C(W) \subseteq C(cl(W))$; thus $W \in C(cl(W))$.

Next, let $Z^* \in C(Y)$. By Lemma 4, $c\ell(Z^*) = c\ell(Y)$. Suppose, for contradiction, that there exists $W \in \min\{Z \subseteq Y : c\ell(Z) = c\ell(Y)\}$ such that $W \subsetneq Z^*$. By our previous argument it follows $W \in C(c\ell(W))$. From the fact that $c\ell(Y) = c\ell(W)$ we know that $C(c\ell(Y)) = C(c\ell(W))$ and therefore that $W \in C(c\ell(Y))$. But by Lemma 2, $C(Y) \subseteq C(c\ell(Y))$ and hence $Z^* \in C(c\ell(Y))$. Because $W \neq Z^*$ this contradicts the assumption that C is non-nested. Therefore, $Z^* \in \min\{Z \subseteq Y : c\ell(Z) = c\ell(Y)\}$, which implies that $C(Y) \subseteq \min\{Z \subseteq Y : c\ell(Z) = c\ell(Y)\}$.

Lastly, let $Z^* \in \min\{Z \subseteq Y : c\ell(Z) = c\ell(Y)\}$. By our prior argument, $Z^* \in C(c\ell(Z^*))$. Because $Z^* \subseteq Y \subseteq c\ell(Z^*)$ and $Z^* \in C(c\ell(Z^*))$ we know from Lemma 5 that $Z^* \in C(Y)$. Therefore, $\min\{Z \subseteq Y : c\ell(Z) = c\ell(Y)\} \subseteq C(Y)$, completing the proof. □

Theorems 3 and 6 yield the following.

Theorem 7. *Suppose that C is a path-independent and non-nested choice correspondence. Then there is a family $\mathcal{W}$ of weak orders for which $C(Y) = \bigcup_{\succeq \in \mathcal{W}} \operatorname*{argmax}_{Y} \succeq$.*

This result establishes that any non-nested and path-independent choice correspondence can be represented as a collection of weak preferences such that the choice from any set is the union of the maximal elements with respect to each preference. This is a generalization of the decomposition result of Aizerman and Malishevski (1981). In a recent work, Chambers and Yenmez (2013) use this decomposition result in the context of matching.

3 Conclusion

We have investigated the relationship between binary relations satisfying certain properties and closure operators. Our results have crucial implications in the theory of incomplete preferences, preference for flexibility in decision theory, and path-independent choice. We plan to find more applications using our results in future work.

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