The Limits of Tolerance

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Abstract

I propose a model of aggregation of intervals relevant to the study of legal standards of tolerance. Seven axioms: responsiveness, anonymity, continuity, strategyproofness, and three variants of neutrality are then used to prove several important results about a new class of aggregation methods called endpoint rules. The class of endpoint rules includes extreme tolerance (allowing anything permitted by anyone) and a form of majoritarianism (the median rule).

1 Introduction

Common law legal systems often rely on community standards, a legal concept according to which actions are judged according to the standards of the society. Community standards are ubiquitous in the common law. In the law of accidents, negligence is determined according to the standard of the reasonable person.[1] In contract law, determinations of good faith are made by reference to "community standards of decency,

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[1] "We apply the standards which guide the great mass of mankind in determining what is proper conduct of an individual under all the circumstances and say that he was or was not justified in doing the act in question." *Osborne v. Montgomery*, 203 Wis. 223, at 231 (1931). There is a considerable debate about the extent to which courts should and do follow this general rule; an alternative approach treats the reasonable person as a normative standard. For more, see Miller and Perry (2012).

fairness, or reasonableness."[2] In the United States, speech that is obscene according to "contemporary community standards" may be criminalized notwithstanding constitutional protection of free speech.[3] These standards are used in cases where absolute standards of behavior are undesirable because the standard is hard to define or expected to change over time.

I introduce a model of community standards in which standards are represented by intervals of the real line. A set of individual standards is aggregated to form a community standard. Normative axioms are introduced that represent principles important in legal decision-making. These axioms are used to characterize a new family of aggregation methods: the *endpoint rules*.

To illustrate, consider the law of defamation, which gives people a right to sue for damages when someone else publishes speech that harms one's reputation in the eyes of the community (see Miller and Perry, 2013b). The person may claim, for example, that the statement attributed to her extreme political views that she does not in fact hold, and that the association with those views has harmed her reputation.[4] Members of society each believe that respectable people are neither on the extreme left nor the extreme right, but they may have different beliefs about what is extreme in each direction. Alternatively, we may think of these cutoffs as defining an interval of reputable political beliefs. The court must determine whether the speech is defamatory, and this determination must be based on the beliefs of the members of the community.

Endpoint rules use these cutoffs—the points beyond which an action is deemed unreasonable, or a statement defamatory—to determine the limits of social tolerance. The formal structure of these rules will be described below, but the basic concept of these rules is that the bounds of tolerance are set by eliminating the most extreme cutoffs. In the context of defamation, this implies that a court will not merely look to see whether individuals consider the speech to be extreme, but rather, whether they agree it to be extreme for the same reasons. The rules eliminate the possibility that the statement harms the plaintiff's reputation because some members of the community consider it to be too far to the left, and others consider it to be too far to the right.

I now proceed to describe the model. The first element of the model is the set of alternatives, which is isomorphic to the real line. Next, there are agents; these represent the members of the community. Each agent has a set of allowable alternatives; this set is assumed to be bounded and convex.[5] In other words, judgments take the form of intervals of the real line. These individual judgments are then aggregated to

[2] Restatement (Second) of Contracts, §205 cmt. a. For more see Miller and Perry (2013a).

[3] *Miller v. California*, 413 U.S. 15 (1973). See also Miller (2013).

[4] There are, of course, many political dimensions; nonetheless, the language of political left and political right suggests that a model with one dimension may have some explanatory power. We may also restrict attention to beliefs along a single dimension, such as views on relating to the use of military force.

[5] To simplify results, I assume that judgments take the form of open sets.

form the community standard.

The focus of this paper is on the method through which these judgments are combined. A set of axioms is defined and used (in various combinations) to characterize the family of endpoint rules. These rules aggregate the lower and upper endpoints separately, in a way that guarantees that that aggregate will be an interval.

Endpoint rules are parametrized by two positive integers, p and q, such that the sum of the two parameters is not greater than one more than the number of agents. As each agent has a judgment that takes the form of an interval, we can define the **p,q**-th endpoint rule as the one that defines the aggregate set to be the interval defined by the p-th lowest lower endpoint and the q-th highest upper endpoint. The subclass of rules where $p = q$ are called "symmetric" endpoint rules.

The family of endpoint rules includes the "maximal rule," in which $p = q = 1$ (see Miller, 2009; Gaba et al., 2017), and the "median rule," in which $p = q = \lfloor \frac{n+1}{2} \rfloor$ (see Block, 2010; Miller, 2009; Farfel and Conitzer, 2011; Gaba et al., 2017). In practice, there may not be a well-defined median judgment. Endpoint rules provide one answer to this problem: as in Lax (2007), the median rule is well-defined even though a median judgment may not exist.

An important feature of endpoint rules is that they aggregate judgments according to their endpoints, and not in a pointwise manner. This is important because pointwise aggregation (for example, according to majority rule) will not necessarily result in a well-defined interval and may, as such, lead to incoherent outcomes. A further feature of these rules is that they aggregate the endpoints independently. That is, the aggregate left endpoint is a determined without reference to the individual right endpoints, and vice versa.

To return to our example, a court may ask a jury to determine whether a statement harms the reputation of the plaintiff in the eyes of the community. The jury may unanimously agree that the statement does in fact harm the plaintiff's reputation. But they may do so for different reasons; some may consider it too liberal, and others, too conservative. As such, it is possible (through unanimous or majority aggregation) that all statements would be judged defamatory. But society cannot function if all statements are considered to be defamatory. Under majority aggregation, we can reach an equally perverse result: it may be possible that calling someone a moderate is defamatory, but calling them a liberal or a conservative is not.

In practice, a court will only consider the case in front of it. As a consequence, an observer may not receive enough information to tell whether similar cases would have been decided in a coherent manner. But the problem of incoherence still remains, and the decision would still be fundamentally arbitrary.

Endpoint rules solve this problem by separating the decision of whether an action is permissible into two questions: First, is there a 'lesser' permissible action? Second, is there a 'greater' permissible action? That is, we first check to see whether enough people believe that there is a statement that could be made, referring to that individual as being further to the left, that would not be defamatory. They need not agree on what that statement would be, but only that each believes that such a statement would exist. We then check to see whether enough people (though not necessarily the same number as before) believe that a statement placing that individual further on the right would be non-defamatory. If the answer to both of these questions is yes, the claim of defamation would fail.

Having described the class of endpoint rules, I characterize it using several axioms. Two are direct analogues of axioms found in Miller (2013). The *responsiveness* axiom requires the aggregation rule to respond to changes in the individual judgments. If the individual interval changes, and each new interval includes the prior one, then responsiveness requires the new aggregate interval to include the prior aggregate interval. The *anonymity* axiom requires the aggregate to be independent of the names of the agents, so that the aggregate choice would not change were two agents to trade their standards between themselves. In addition, I add a *continuity* axiom.

The fourth and fifth axioms are weaker versions of the neutrality axiom used in Miller (2013). To understand these axioms, it is important to recall that the model imposes a structure on the set of alternatives, in that the set of alternatives is isomorphic to the real line. These axioms are motivated by the idea that points on the political spectrum have no inherent specialness. The political center is a function of individual beliefs and cannot be objectively defined; one may note that views on civil rights, same-sex marriage, capital punishment, gun control, and democracy have changed over time. These views also have no meaningful cardinal relationship.

Nonetheless, the political spectrum has a natural structure; I will focus on two important properties of this structure. The first property is *betweenness*; on the political spectrum, a centrist is objectively in between a liberal or a conservative (see Nehring and Puppe, 2002, 2007, for more on betweenness). The second property is *direction*; in some contexts we may wish to treat the left differently from the right. Such a property may be relevant in other contexts, such as when trying to determine the reasonableness of highway driving speeds in negligence.[6] Here, the directions of "high" and "low" are not interchangeable.

The *weak neutrality* axiom requires the aggregation of individual judgments to be independent of transformations of the real line that preserve both betweenness and the direction. It does not require the aggregation to be independent of transformations that preserve betweenness only. This property implies that the cardinal properties of the real line should be disregarded. Weak neutrality is equivalent to the ordinal covariance axiom of Chambers (2007).

Strong neutrality, as its name implies, is stronger than weak neutrality. It requires the aggregation of individual judgments to be independent of any betweenness-preserving transformation of the real line. This property implies that betweenness is important, but that both the direction and cardinal properties of the real line should

[6]In the context of speeds, we may think of the endpoints as being drawn from $(0, \infty]$, to reflect the idea that one should not park in the middle of the highway. Note that, while speeds themselves have a cardinal relationship, the reasonableness of speeds may not.

be disregarded.

Using these axioms, I prove two results. First, I show that the family of endpoint rules is characterized by the responsiveness, anonymity, continuity, and weak neutrality axioms. Second, replacing weak neutrality with strong neutrality yields a characterization of the symmetric endpoint rules.

1.1 Strategyproofness

The interval aggregation problem described above does not use the concept of preference. Individuals' judgments represent the individuals' beliefs about which actions are acceptable, and not their preferences over policy (for more on this distinction, see Kornhauser and Sager, 1986). The model takes the judgments as given, and does not ask where they come from.

However, even though we may be opposed to strategic judgments in some contexts as a matter of principle, this does not mean that strategic judgments are never made. For this reason, one may wish to know the extent to which the endpoint rules are manipulable. To answer this question, I investigate the implications of a *strategyproofness* assumption (see Dummett and Farquharson, 1961).

To study the implications of strategyproofness it is necessary to restrict the class of allowable preferences (see Gibbard, 1973; Satterthwaite, 1975), and it is known that, when choosing a single alternative from a single issue dimension, a voting rule can be strategyproof and non-dictatorial if preferences are single-peaked (see Moulin, 1980). Block (2010) and Farfel and Conitzer (2011) define a class of generalized single-peaked preferences according to which an interval is defined to be between two other intervals if its lower endpoint is between the lower endpoints of the other two, and if its upper endpoint is between the upper endpoints of the other two. Preferences are generalized single-peaked if there is (a) a unique interval that is preferred to all other intervals (called the "peak") and (b) any interval in between the peak and a third interval is necessarily preferred to the third interval (see Nehring and Puppe, 2008). An aggregation rule is strategyproof if each individual prefers to truthfully reveal his or her peak interval. Block (2010) and Farfel and Conitzer (2011) show that the median rule is strategyproof.

I provide a full characterization of anonymous and strategyproof interval aggregation rules. To prove this characterization I first show that the strategyproofness axiom implies that the aggregation rule must aggregate the endpoints independently. As a consequence, the aggregation of lower endpoints is essentially equivalent to the aggregation of single peaked preferences on a single issue dimension as in Moulin (1980). I then show that the results in that work can be used to characterize an a family of rules analogous to Moulin's famous "phantom voters" characterization.

This family includes the endpoint rules as a special case. I show that endpoint rules can be characterized by further adding a *translation equivariance* axiom, which requires the aggregation rule to shift the aggregate interval by a constant when each individual interval shifts by that same constant. Translation equivariance is implied by both the of the neutrality axioms; in this sense it may be thought of as a very weak form of neutrality.

An implication of this axiom is that neutrality is not necessary to support the use of endpoint rules. The translation equivariance is much weaker than neutrality; for example, it allows the rule to make use of the cardinality properties of the real line. Averaging rules—that is, methods through which the endpoints are determined by the use of averages—are translation equivariant and, in some cases, anonymous. But averaging rules are not endpoint rules, and are ruled out by the assumption of strategyproofness.

1.2 Other literature

There is a significant literature devoted to the study of opinion and judgment aggregation, starting with the pioneering works of Arrow (1963) and May (1952). The closest work in this literature is Miller (2013), which differs in that standards in that work are arbitrary subsets of an unstructured set of alternatives, rather than intervals of the real line. The lack of an objective order leads to a near-impossibility result; Miller (2013) provides conditions under which an aggregate of individual standards will deem an action impermissible only when all individuals in the community consider it to be impermissible.[7] As communities are generally understood to be large and diverse, nothing, in practice, would be forbidden. By contrast, the present work provides insight into how the structure of the real line could be exploited to construct more useful rules.

There is a conceptual link between the results in this paper and those of Chambers (2007), which characterizes quantile representations using ordinal covariance and monotonicity. Ordinal covariance is essentially weak neutrality, while monotonicity is closely related to responsiveness. Endpoint rules can be thought of as a type of a quantile rule, where each endpoint is chosen according to a quantile. A contribution of the present work is that endpoint rules allow for independent and consistent aggregation of the two endpoints.

The results involving strategyproofness are related to those in Barberà et al. (1991), which studies a model in which individuals much choose a subset of objects from some finite set. In particular, they use anonymity, neutrality, strategyproofness, and voter sovereignty properties to characterize a mechanism known as "voting by quota," in which objects are chosen if a quota is met; that is, if large enough group of individuals wants those objects to be chosen. The main difference with the results in Barberà et al. (1991) is that, in this work, the objects are ordered, and individuals must choose convex subsets. As a consequence, there is no need for all objects, or even

[7] Ahn and Chambers (2010) reaches a similar result in the context of menu choice. Related results, using different sets of axioms, can also be derived from Monjardet (1990) and Nehring and Puppe (2007).

any object, to be supported by a quota, as an object will be chosen as long as enough people agree that (a) some lesser object that should be chosen and (b) some greater object that should be chosen as well. They need not agree on the identity of the lesser or greater object, but only that some such object exists. The axioms are different; the neutrality axiom in this paper does not apply to all permutations, but instead preserves some of the features of the real line. In addition, the strategyproofness axiom relies on a different assumption regarding preferences.

The results involving strategyproofness are also related to those in Border and Jordan (1983) and Barberà et al. (1998), which characterize strategyproof voting mechanisms when alternatives are subsets of some Euclidean space. The relationship becomes apparent when one considers that intervals may be described as those elements $(x, y) \in \mathbb{R}^2$ for which $x < y$. In this context, the preferences defined in those works imply the generalized single-peaked preferences studied in this paper.

2 Endpoint Rules

Let $N \equiv \{1, \ldots, n\}$ be a finite set of agents, and let Σ be the set of bounded open intervals of the real line. I study aggregation functions $f : \Sigma^N \to \Sigma$, which map a set of n intervals into a single interval. The responsiveness and anonymity axioms used in Miller (2013) can be defined in this environment.

Responsiveness: For all $S, T \in \Sigma^N$, if $S_i \subseteq T_i$ for all $i \in N$, then $f(S) \subseteq f(T)$.

Let π denote a permutation of N, and define $\pi S = \big(S_{\pi(1)}, \ldots, S_{\pi(n)}\big)$.

Anonymity: For every π of N and $S \in \Sigma^N$, $f(S) = f(\pi S)$.

I introduce a basic continuity axiom. Let Σ be the set of open bounded intervals of the real line. For x, y in $\mathbb{R}$, $x < y$, let $W(x, y) \subset \Sigma$ such that (x', y') in $W(x, y)$ if $x' > x$ and $y' < y$. Let (Σ, τ) be the topological space with the base $\cup_x \cup_{y>x} W(x, y)$. Let (Σ^N, τ^N) be the product topology.

Continuity: For all $Y \in \tau$, $f^{-1}(Y) \in \tau^N$.

The neutrality axiom used in Miller (2013) must be modified to take into account of the natural structure. I provide two distinct neutrality axioms. Let Φ be the set of all strictly monotone transformations of the real line, and let Φ^+ be the set of all strictly increasing monotone transformations of the real line. That is, transformations in Φ must preserve betweenness, while transformations in Φ^+ must additionally preserve the direction. In neither set, however, are transformations required to preserve the cardinal properties of the real line. For $S_i \in \Sigma$ and $\phi \in \Phi$, define $\phi(S_i) \equiv \cup_{x \in S_i} \phi(x)$.

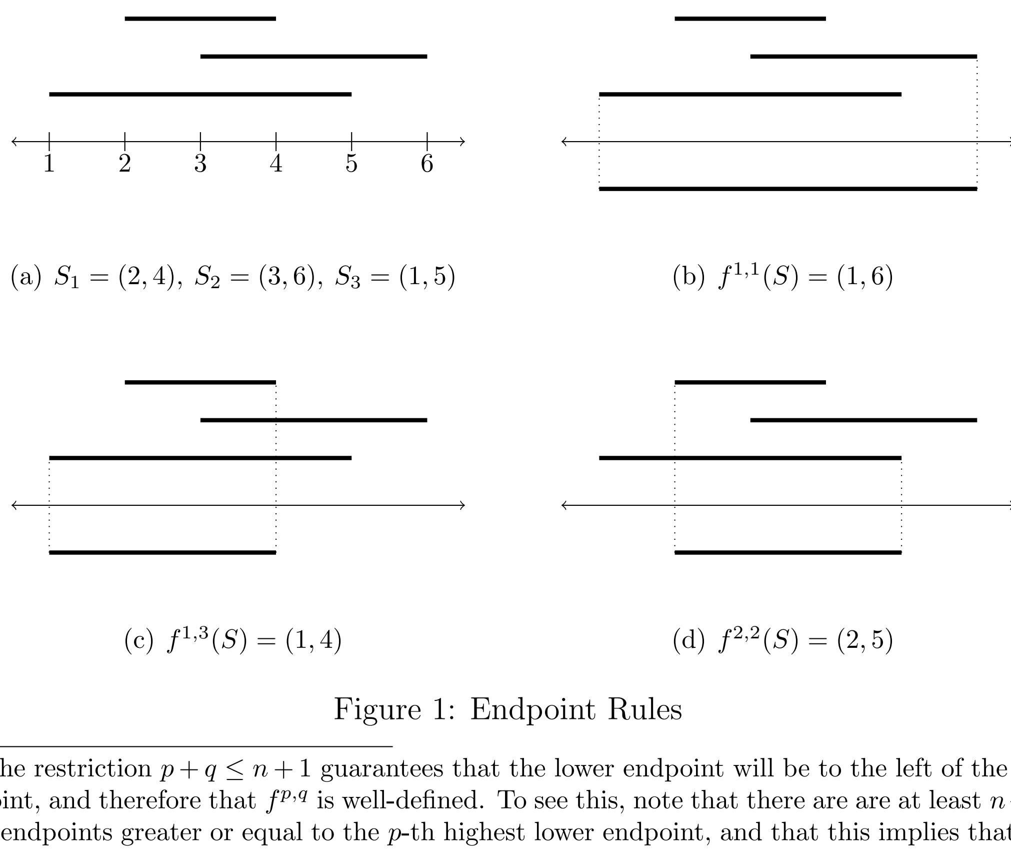
Weak Neutrality: For every $\phi \in \Phi^+$ and $S \in \Sigma^N$, $\phi(f(S)) = f(\phi(S_1), \ldots, \phi(S_n))$.

Strong Neutrality: For every $\phi \in \Phi$ and $S \in \Sigma^N$, $\phi(f(S)) = f(\phi(S_1), \ldots, \phi(S_n))$.

I introduce a class of aggregation rules, called endpoint rules, which I believe have not yet been described in the literature. This class of rules is parameterized by two positive integers, p and q, such that $p + q \leq n + 1$. For a profile of standards S and a point $x \in \mathbb{R}$, let $G^+(S, x) = \{i \in N : (-\infty, x] \cap S_i \neq \varnothing\}$ be the set of individuals who have x, or a point below x, in their interval, and let $G^-(S, x) = \{i \in N : [x, +\infty) \cap S_i \neq \varnothing\}$ who have x, or a point above x, in their interval. An endpoint rule is of the form

$$f^{p,q}(S) \equiv \{x : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}.$$

This rule takes the open interval defined by the p-th lowest lower endpoint and the q-th highest upper endpoint.[8] For example, three individuals are depicted in Figure 1(a), with standards $S_1 = (2, 4)$, $S_2 = (3, 6)$, and $S_3 = (1, 5)$. Here, $f^{1,1}(S) = (1, 6)$ (Figure 1(b)), $f^{1,3}(S) = (1, 4)$ (Figure 1(c)), and $f^{2,2}(S) = (2, 5)$ (Figure 1(d)).



(a) $S_1 = (2, 4)$, $S_2 = (3, 6)$, $S_3 = (1, 5)$

(b) $f^{1,1}(S) = (1, 6)$

(c) $f^{1,3}(S) = (1, 4)$

(d) $f^{2,2}(S) = (2, 5)$

Figure 1: Endpoint Rules

[8]The restriction $p + q \leq n + 1$ guarantees that the lower endpoint will be to the left of the upper endpoint, and therefore that $f^{p,q}$ is well-defined. To see this, note that there are are at least $n - p + 1$ lower endpoints greater or equal to the p-th highest lower endpoint, and that this implies that there are at least $n - p + 1$ upper endpoints greater than the p-th highest lower endpoint. Consequently the rule is well-defined whenever $q \leq n - p + 1$, or when $p + q \leq n + 1$.

An important subclass of rules is that of the symmetric endpoint rules, where $p = q$. If we define m to be the maximal integer less or equal to $(n+1)/2$, then $f^{m,m}(S)$ is the median rule (see Block, 2010; Miller, 2009; Farfel and Conitzer, 2011; Gaba et al., 2017).

I present two results. The first theorem is a characterization of the endpoint rules.

Theorem 1. *An aggregation rule f satisfies responsiveness, anonymity, continuity, and weak neutrality if and only if it is an endpoint rule.*

The second theorem is a characterization of symmetric endpoint rules.

Theorem 2. *An aggregation rule f satisfies responsiveness, anonymity, continuity, and strong neutrality if and only if it is a symmetric endpoint rule.*

The proofs are in the appendix. The sets of axioms used in both theorems are independent when $n \geq 3$; however, symmetric endpoint rules can be characterized without continuity in the case where $n = 2$.[9]

3 Strategyproofness

To study the question of strategyproofness it is necessary to make an assumption about preferences.[10] Block (2010) and Farfel and Conitzer (2011) define a class of single-peaked preferences on intervals that relies on a concept of betweenness. An interval is defined to be between two other intervals if its lower endpoint is between the lower endpoints of the other two, and if its upper endpoint is between the upper endpoints of the other two. For intervals $R, T \in \Sigma$, let $\mathcal{B}(R,T) \subset \Sigma$ be the set such that $S \in \mathcal{B}(R,T)$ if

$$\inf R \leq \inf S \leq \inf T \textbf{ or } \inf R \geq \inf S \geq \inf T \tag{1}$$

and

$$\sup R \leq \sup S \leq \sup T \textbf{ or } \sup R \geq \sup S \geq \sup T. \tag{2}$$

Single-peaked preferences are preferences for which (1) there is a unique preferred interval (the 'peak') and (2) an interval that is between the peak and a third interval is preferred to that third interval. Let $\mathcal{P}$ be the set of preferences on Σ such that, for all $\succeq_i \in \mathcal{P}$, (1) there exists $S_i^* \in \Sigma$ such that $T \succeq_i S_i^*$ implies that $T = S_i^*$ and (2) for $R, T \in \Sigma$, $R \in \mathcal{B}(S_i^*, T)$ implies that $S_i^* \succeq_i R \succeq_i T$.

An aggregation rule is strategyproof if it is always in an agent's interest to reveal her preferred interval, holding the other judgments constant.

[9]The proofs of these facts are left as an exercise for the reader.

[10]No assumption about preferences has been made up to this point, as judgments need not come from a preference ordering.

Strategyproofness: For every $S \in \Sigma^N$, $i \in N$, and $\succeq_i \in \mathcal{P}$, $f(S_i^*, S_{-i}) \succeq_i f(S)$.

Block (2010) and Farfel and Conitzer (2011) show that the median rule is strategyproof. It is straightforward to show that endpoint rules are strategyproof as well.[11]

I provide a complete characterization of anonymous and strategyproof rules.

Endpoint rules aggregate the upper and lower endpoints independently; that is, these rules aggregate the lower endpoints without considering the upper endpoints, and vice versa. The property can be defined formally as follows:

Independent aggregation of endpoints: For every $S, T \in \Sigma^N$, (a) $\inf S_i = \inf T_i$ for all $i \in N$ implies that $\inf f(S) = \inf f(T)$ and (b) $\sup S_i = \sup T_i$ for all $i \in N$ implies that $\sup f(S) = \sup f(T)$.

I show that all strategyproof rules have this property.

Proposition 1. *An aggregation rule f satisfies strategyproofness only if it satisfies independent aggregation of endpoints.*

The proof of this proposition is in the appendix.

A consequence of Proposition 1 is that, under the assumption of strategyproofness, the aggregation of lower (and upper) endpoints is essentially equivalent to the aggregation of single-peaked preferences on a single issue dimension, as studied in Moulin (1980). This allows for an analogue of Moulin's characterization of "phantom voters" that includes the endpoint rules as a special case.

Define $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, \infty\}$ as the extended reals, and define $\overline{<}$ as a binary relation on $\overline{\mathbb{R}}$ such that, for $x, y \in \overline{\mathbb{R}}$, $x \overline{<} y$ if (i) $x, y \in \mathbb{R}$ and $x < y$, (ii) $x = -\infty$, or (iii) $y = \infty$. Let $\overline{\Sigma} = \{(x, y) \in \overline{\mathbb{R}} : x \overline{<} y\}$ be the open convex intervals of the extended reals. Note that $\Sigma \subset \overline{\Sigma}$. Let $\text{MED} : \overline{\Sigma}^{2n+1} \to \overline{\Sigma}$ be the median rule applied to $2n+1$ intervals of the extended reals; that is, for a profile $Q \in \overline{\Sigma}^{2n+1}$, $\text{MED}(Q) = \{x \in \mathbb{R} : \min\{|\{i : (-\infty, x] \cap Q_i \neq \varnothing\}|, |\{i : [x, +\infty) \cap Q_i \neq \varnothing\}|\} \geq n+1\}$.

The following claim characterizes the class of anonymous and strategyproof rules. The proof relies on Moulin (1980) and is found in the appendix.

Claim 1. *An aggregation rule f satisfies anonymity and strategyproofness if and only if there exists $P \in \overline{\Sigma}^{n+1}$ such that $f(S) = \text{MED}(S, P)$ for all $S \in \Sigma^N$.*

Here, the elements $P \in \overline{\Sigma}$ may be thought of as "phantom intervals;" unlike the regular intervals, however, these include the half-bounded intervals of the form $(-\infty, x)$ and (x, ∞) for $x \in \mathbb{R}$, and the fully unbounded intervals of the form $(-\infty, -\infty)$, $(-\infty, \infty)$, and (∞, ∞).[12]

[11] In a finite setting, strategyproofness of endpoint rules can be proven using Nehring and Puppe (2008, Theorem 3).

[12] Alternatively, one may think of these as "calibration intervals," following Thomson (2018).

The weak neutrality axiom would further imply that the resulting rules are endpoint rules. This is because weak neutrality would eliminate the fully bounded and half-bounded intervals; thus all phantom intervals would be fully unbounded. The connection can be easily observed: the p, q-th endpoint rule is characterized by $p \geq 1$ phantoms of the form (∞, ∞), $q \geq 1$ phantoms of the form $(-\infty, -\infty)$, and $n+1-p-q$ phantoms of the form $(-\infty, \infty)$.[13]

However, endpoint rules can be characterized without the full strength of the neutrality axiom. A much weaker axiom, translation equivariance, is sufficient to reach the same result. For $b \in \mathbb{R}$ and $S_i \in \Sigma$, define $[S+b]_i = (\inf S_i + b, \sup S_i + b)$ to be the interval S_i shifted by b. For $S \in \Sigma^N$, define $[S+b] = ([S+b]_1, \cdots, [S+b]_n)$.

Translation equivariance: For every $S \in \Sigma^N$ and $b \in \mathbb{R}$, $f([S+b]) = [f(S)+b]$.

The endpoint rules are characterized by strategyproofness, anonymity, and translation equivariance.

Theorem 3. *An aggregation rule f satisfies anonymity, strategyproofness, and translation equivariance if and only if it is an endpoint rule.*

The proof is in the appendix.

4 Other applications

The model is motivated by legal rules according to which actions are judged by reference to community standards of behavior. However, the aggregation of intervals has been studied elsewhere in economics. To varying extents, the results reported in this paper may be applicable to these other contexts.

A first type of application involves a panel of experts that needs to provide an aggregate opinion, where the expert opinions take the form of a range. Farfel and Conitzer (2011) provides an example of climatologists who must aggregate their views about the likely range of global warming outcomes, when distrust hampers their ability to report anything more than a range of possibilities. These expert opinions may be in the form of a range because the estimate can be limited by information constraints, and because the outcome can depend on choices made by future policy makers. (See Chatfield, 1993, for additional reasons why experts may make interval forecasts.)

A second type of application involves a group of principals that needs to delegate a decision to a better informed agent. A panel of judges may need to agree on a sentence that takes the form a range (e.g. five-to-ten years) so that a parole board can base the ultimate sentence on the convict's subsequent behavior. Or a legislature may

[13] In addition, strong neutrality would imply that the number of intervals of the form $(-\infty, -\infty)$ must equal the number of the form (∞, ∞), and therefore that $p = q$.

wish to choose a sentencing range for certain crimes, so as to constrain future judges (Farfel and Conitzer, 2011). A significant literature in political science addresses problem of bureaucratic drift, or how administrative agency decisions can deviate away from those preferred by the legislature (see Epstein and O'Halloran, 1994, 1996, 1999, 2008; Huber and Shipan, 2002, 2006; Gailmard, 2009; Gailmard and Patty, 2012; Callander and Krehbiel, 2014). This literature primarily follows the model of delegated delegation found in the principal-agent literature (see Holmström, 1977, 1984; Melumad and Shibano, 1991; Alonso and Matouschek, 2008; Kleiner, 2022), in which delegation commonly takes the form of an interval of permissible alternatives.

A difficulty in this last literature is that, as legislatures are not unitary entities, the identification the legislature with a preference is problematic for reasons first identified by Arrow (1963). The papers in this literature tend to assume away the difficulty of combining legislators' preferences by making strong assumptions; for example, the existence of a median legislator (see, for example Epstein and O'Halloran, 1994). Having done so, these papers can first aggregate the preferences (to that of the median legislator) and then find the legislature's preferred interval. However, rather than make strong assumptions on the legislators' preferences, one could reverse the order of these steps—to first find each individual legislator's preferred interval, and then to aggregate these to find the legislature's interval.

A third type of application involves a group of countries trying to determine the terms of a multi-lateral treaty. A trade agreement may place bounds on the level of permissible tariffs (see Kleiner, 2022). A defense treaty may place limits on the level of military spending. Members of the military pact may want their members to commit to spending a certain amount of money each year (to avoid free riding) but may also want to limit the maximum amount that their partners spend (because of distrust).

To what extent are the model and the axioms applicable to our understanding of these related problems? I assess these each in turn.

First, to the what extent does the model apply in these other contexts? Community standards are based on judgments, and not preferences. That is, courts judge the permissibility of actions according to whether the action is considered acceptable in the community, and not according to whether the majority would like it to be allowed. These individual judgments are not necessarily the result of preferences over alternatives or over judgments. An individual may judge an act to be permitted despite a preference that it be banned, or vice versa.

In other contexts, however, judgments may arise as a predictable result of underlying preferences and information. For example, a panel of experts that needs to provide a collective judgment is primarily tasked with aggregating information. But because of different attitudes toward risk, or different discounting rates, their preferences may also differ. Judges, legislators, and governments similarly differ in their preferences and information. For this reason, one might build a model that starts with preferences and information, rather than the resultant judgments.

There are, though, some problems with aggregating the preference and information. The first is that it may be difficult to incentivize the actors to report their preferences and information truthfully. For this reason Farfel and Conitzer (2011) suggests that distrust may lead agents to limit their reporting to an interval.

When preferences can be observed, there is the general difficulty of aggregating preferences in a meaningful way. Even if it is possible to aggregate the preferences (for example, if the preferences take the form of von Neumann-Morgenstern utility functions), there is an added difficulty of simultaneously aggregating preferences and information (see Hylland and Zeckhauser, 1979). Aggregating the judgments directly will not solve all problems, but may in some cases simplify the task.

A model of preferences and information, however, can be useful even when direct aggregation is impossible. If only information is to be aggregated, then the existence of a "correct" answer makes it possible to study methods in terms of the quality of their results (see, for example Yaniv, 1997; Gaba et al., 2017). Furthermore, the model assumes a full domain of intervals. But, if intervals come from preferences or information, it is possible that some intervals will never arise in practice.

Second, to what extent are the axioms meaningful when applied to other contexts? As the continuity axiom is a technical condition, I discuss the relevance of the remaining axioms—anonymity, responsiveness, neutrality, and strategyproofness—to these alternative applications.

Anonymity is important in context of community standards. To the extent that individuals are members of the relevant community, their views are generally treated equally. Similarly, members of legislatures, judges on a court, and independent states are generally presumed to be equal.

Nonetheless, there are at least two reasons why anonymity may at times be undesirable. The first is if the judgments come from individuals with different levels of expertise. One might want to include a graduate student on an expert panel but nonetheless treat that student differently from a Nobel laureate. This may also apply in select cases of community standards, such as a "reasonable doctor" standard that relies on the doctors' expertise.

The second reason is that some parties may have unequal bargaining power or unequal legal rights. For example, consider the case of treaty negotiations. In principle, all states are equal.[14] Nonetheless, this principle has exceptions,[15] and even countries with equal rights may agree to a non-anonymous rule due to the secure the participation of a more powerful state in treaty negotiations.

The responsiveness axiom is important in the context of community standards. No individual should be punished because of an increase in the bounds of tolerance.

[14] "The Organization is based on the principle of the sovereign equality of all its Members." U.N. Charter art. 2, para. 1.

[15] The UN security council is a notable case, as it has five permanent members (United States, United Kingdom, France, Russia, and China) with veto powers of security council decisions. U.N. Charter art. 23, para 1; art. 27, para 3. However, the security council does not negotiate treaties.

The value of this axiom will otherwise vary across cases. It will generally be desirable if the intervals arise from preferences. However, this is less clear if the intervals arise from information. An expert who provides too wide of a range may be considered to be useless, and ignored. Alternatively, though, an expert who provides a wide range may be providing valuable information—that in the expert's eyes, we know very little—and if so, a responsive aggregation rule would incorporate that information.

The neutrality axioms require the aggregation rule to disregard the cardinal properties of the real line. This suggests that the neutrality axioms are relevant when aggregating qualitative, but not quantitative, standards. For example, a community standard of offensiveness is likely to be qualitative—one may be able to order works in terms of offensiveness—but there may be no objective way to make cardinal comparisons. In contrast, a range of temperatures or probabilities is quantitative.

In some contexts it can be hard to determine whether standards are qualitative or qualitative. A court may need to determine whether a particular speed was reasonable. But because there is no objective cardinal mapping between the speeds and the extent to which they are reasonable, that court may choose to disregard the cardinal relationship between the possible speeds.

The strategyproofness axiom is important in cases in which individuals have preferences over intervals, and in which those preferences satisfy the generalized single-peakedness condition introduced by Block (2010) and Farfel and Conitzer (2011) and followed in this paper. Legislators who delegate discretion to bureaucratic agencies have preferences over the delegated interval. It is possible that these preferences are generalized single-peaked, though this remains to be shown.

In the context of community standards, the strategyproofness axiom is mostly irrelevant, as individuals do not directly report their standards. In some cases, however, it may be possible for individuals to change their standards with the aim that it will affect a future community standard.[16]

The neutrality and strategyproofness axioms exclude the possibility of averaging rules, in which aggregate endpoints are averages of individual endpoints. Neutrality is a reasonable justification in the contexts in which averaging does not make much practical sense—for example, when taking a numerical average would not be particularly meaningful. However, even in cases where averaging makes sense on its face, strategyproofness implies that it should not be used in cases where the parties are interested in the outcome and have some ability to misrepresent their information or preferences.

[16]For example, states are often concerned with how their stated standards will, in the long run, affect the creation of customary international law.

5 Conclusion

I have introduced the endpoint rules and have shown that they are characterized by responsiveness, anonymity, continuity, and neutrality in this setting. Furthermore, I have shown that with a suitable restriction on preferences, endpoint rules are strategyproof, and that all strategyproof, anonymous, and neutral aggregation rules are endpoint rules.

One may ask whether more general results can be established by focusing on the abstract properties of the betweenness relation and the order. To provide a short example: consider the case of a decision that must be made on two (or more) dimensions. We must decide not only how fast it is reasonable to drive, but also, how much training drivers should have before getting behind the wheel. There may be a tradeoff; at higher speeds, more training is necessary, although different individuals may have different views about the right tradeoff. This is a much more complex question, and it is difficult to study. In the case of the real line, the concept of betweenness implies intervals, which can be identified with points in two-dimensional space (such that $x_1 < x_2$). In the case of multidimensional space, however, betweenness simply implies convexity, and there is no similarly easy way to represent these convex sets. In addition, there may be interesting problems with different underlying structures, for which the simple assumption of Euclidean space may not be applicable.

Future research may investigate the relationship between community standards and other economic problems that can be modeled through interval aggregation. Section 4 provides reasons to think that the insights in this paper may be applicable in some of these cases. However, the legal problem that motivates this work is different from those posed in these other contexts. Individual standards are subjective and can be motivated by abstract concerns, or no concerns at all. Community standards do not necessarily exist to serve a consequentialist goal.[17] This is not necessarily the case in these other economic environments, and the conclusions of this paper should not be applied elsewhere without a careful understanding of the underlying problems.

Appendix

I first state and prove the following lemma.

Lemma 1. *If f satisfies anonymity and weak neutrality, then for every $S, T \in \Sigma^N$, every permutation π of N, and every $\phi \in \Phi^+$ such that $\pi S = \phi T$, if there is an endpoint rule $f^{p,q}$ such that $f(S) = f^{p,q}(S)$, then $f(T) = f^{p,q}(T)$.*

Proof of Lemma 1. Let $S, T \in \Sigma^N$, and let π be a permutation of N and $\phi \in \Phi^+$ such that $\pi S = \phi T$. Let f satisfy anonymity and weak neutrality, and let $f^{p,q}$ be an

[17] While the reasonable person standard may exist to reduce the cost of accidents, this is not a universally accepted goal. In the context of obscenity, offense to community standards is often the justification (and not merely the test) for criminal prosecution.

endpoint rule such that $f(S) = f^{p,q}(S)$. Note that by the definition of the endpoint rule, $f^{p,q}(S) = \phi f^{p,q}(T)$. By anonymity, $f(S) = f(\pi S) = f(\phi T)$. By weak neutrality, $f(\phi T) = \phi f(T)$, and therefore $f(S) = \phi f(T)$. Because ϕ is strictly monotone there exists an inverse $\phi^{-1} \in \Phi$ such that $\phi^{-1}\phi S = S$; therefore $\phi^{-1} f(S) = f(T)$ and $\phi^{-1} f^{p,q}(S) = f^{p,q}(T)$. Because $f(S) = f^{p,q}(S)$ it follows that $f(T) = f^{p,q}(T)$. □

Proof of Theorem 1. That endpoint rules satisfy the four axioms is trivial. Let f satisfy the four axioms. I show that f must be an endpoint rule. For $S \in \Sigma^N$ and $p, q \leq n$, define the function $f^{p,q}(S) \equiv \{x : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}$, and define $Q(S) \equiv \{(p, q) \in N^2 : f(S) = f^{p,q}(S)\}$. I will show that there exists $p, q \leq n$, where $p + q \leq n + 1$, such that $(p, q) \in Q(S)$ for all $S \in \Sigma^N$.

Part One: I show that $|Q(S)| \geq 1$ for all $S \in \Sigma^N$.

For $S \in \Sigma^N$ define $L(S) \equiv \cup_i \inf S_i$ and $U(S) \equiv \cup_i \sup S_i$. It is sufficient to show that $\inf f(S) \in L(S)$ and $\sup f(S) \in U(S)$.

First, I show that for all $S \in \Sigma^N$, $\inf f(S), \sup f(S) \in L(S) \cup U(S)$. To see this, let $S \in \Sigma^N$ and suppose, contrariwise, that $\inf f(S) \notin L(S) \cup U(S)$. Let $\phi \in \Phi^+$ such that $\phi(\inf f(S)) \neq \inf f(S)$ and, for all $i \in N$, $\phi(\inf S_i) = \inf S_i$ and $\phi(\sup S_i) = \sup S_i$. Then $S = \phi S$, so $f(S) = f(\phi S)$. By weak neutrality, $f(\phi S) = \phi(f(S))$ and therefore, $f(S) = \phi f(S)$. It follows that $\inf f(S) = \inf(\phi f(S)) = \phi(\inf f(S))$, a contradiction.

Next, I show that for all $S \in \Sigma^N$, $\inf f(S) \in L(S)$ and $\sup f(S) \in U(S)$. Suppose, contrariwise, that this is false, and assume, without loss of generality, that $\inf f(S) \notin L(S)$. Because $\inf f(S) \notin L(S)$, it must be that $\inf f(S) \in U(S)$. Therefore, there exists a group $M \subseteq N$, $M \neq \varnothing$, such that $\inf f(S) = \sup S_j$ for all $j \in M$.

Let $\varepsilon > 0$ such that, for all $i \in N$, $\inf f(S) \geq \inf S_i$ if and only if $\inf f(S) + \varepsilon \geq \inf S_i$, and, for all $j \in N \setminus M$, $\inf f(S) \geq \sup S_i$ if and only if $\inf f(S) + \varepsilon \geq \sup S_i$.

Let $\phi \in \Phi^+$ such that (i) for all $i \in N$, $\phi(\inf S_i) = \inf S_i$, (ii) for all $j \in N \setminus M$, $\phi(\sup S_j) = \sup S_j$, and (iii) $\phi(\inf f(S)) = \inf f(S) + \varepsilon$.

Let $S' \in \Sigma^N$ such that, for all $j \in N \setminus M$, $S'_j = S_j$ and, for all $k \in M$, $S'_k = (\inf S_k, \sup S_k + \varepsilon)$. Because $S_i \subseteq S'_i$ for all $i \in N$ it follows that $f(S) \subseteq f(S')$, and therefore that $\inf f(S) \geq \inf f(S')$. Because $\phi S = S'$ it follows that $f(\phi S) = f(S')$, and by responsiveness that $f(\phi S) = \phi f(S) = \phi f(S)$. Hence $\phi f(S) = f(S')$, and therefore $\inf f(S) + \varepsilon = \inf f(S')$. This implies that $\inf f(S) < \inf f(S')$, a contradiction.

Part Two. For $k \in N$, let $S^k \in \Sigma^N$ such that, for all $i < k$, $S^k_i = (2i - 1, 2i)$, and for all $j \geq k$, $S^k_j = (j + k - 1, j + n)$. I prove that $Q(S^1) = Q(S^n)$. It is sufficient to show that $Q(S^k) = Q(S^{k+1})$ for all $k \in N \setminus \{n\}$.

Let $k \in N \setminus \{n\}$. From Part One we know that $|Q(S^k)| \geq 1$. Because $f^{p,q}(S^k) = f^{p',q'}(S^k)$ only if $p' = p$ and $q' = q$, it follows that $|Q(S^k)| \leq 1$. Thus, $|Q(S^k)| = 1$.

For $\ell \in \{1, \ldots, n-k\}$, let $T^{\ell,+}, T^{\ell}, T^{\ell,-} \in \Sigma^N$ such that $T^{\ell,+}_k = (2k-1, n+k-\ell+\varepsilon)$, $T^{\ell}_k = (2k - 1, n + k - \ell)$, $T^{\ell,-}_k = (2k - 1, n + k - \ell - \varepsilon)$, and for $i \neq k$, $T^{\ell,+}_i = T^{\ell}_i = T^{\ell,-}_i = S^k_i$.

First, I show that for $\ell \in \{1, \ldots, n - k\}$, $Q(T^{\ell,+}) = Q(T^{\ell,-})$. To see this, let

$Y = \{(a,b) \in \Sigma : a > \inf T^\ell - 0.1 \text{ and } b < \sup T^\ell + 0.1\}$. Note that $Y \in \tau$ and that $f(T^\ell) \in Y$. By continuity, $T^\ell \in f^{-1}(Y) \in \tau^N$. It follows that there exists ε such that $T^{\ell,+}, T^{\ell,-} \in f^{-1}(Y) \in \tau^N$ and therefore that $f(T^{\ell,+}), f(T^{\ell,-}) \in Y$. This implies that (a) $Q(T^{\ell,+}) = Q(T^{\ell,-})$.

Next, let $\phi^0 \in \Phi^+$ such that (i) $\phi^0(n+k) = n+k-1+\varepsilon$ and (ii) for all $x \in \mathbb{N} \setminus \{n+k\}$, $\phi^0(x) = x$. Because $\phi^0 S^k = T^{1,+}$, it follows as a consequence of Lemma 1 that (b) $Q(S^k) = Q(T^{1,+})$.

For $\ell \in \{1, \ldots, n-k-1\}$, let $\phi^\ell \in \Phi^+$ such that (i) $\phi^\ell(n+k-\ell-\varepsilon) = n+k-\ell-1+\varepsilon$ and (ii) for all $x \in \mathbb{N}$, $\phi^\ell(x) = x$. Because $\phi^\ell T^{\ell,-} = T^{\ell+1,+}$, it follows as a consequence of Lemma 1 that (c) $Q(T^{\ell,-}) = Q(T^{\ell+1,+})$.

Let $\phi^{n-k} \in \Phi^+$ such that (i) $\phi^{n-k}(2k-\varepsilon) = 2k$, (ii) for all $i \in \{k+1, \ldots, n\}$, $\phi^{n-k}(i+k-1) = i+k$, and (iii) for all $x \in \mathbb{N} \setminus \{2k, \ldots, n+k\}$, $\phi^{n-k}(x) = x$. Because $\phi^{n-k} T^{n-k,-} = S^{k+1}$, it follows as a consequence of Lemma 1 that (d) $Q(T^{n-k,-}) = Q(S^{k+1})$.

By combining (a), (b), (c), and (d), we have that $Q(S^k) = Q(S^{k+1})$.

Part Three. Let $\dot{p}, \dot{q} \leq n$ such that $f(S^1) = f^{\dot{p},\dot{q}}(S^1)$. I prove that for all $S \in \Sigma^N$, $f(S) = f^{\dot{p},\dot{q}}(S)$.

First, I show that $f(S) \subseteq f^{\dot{p},\dot{q}}(S)$. Suppose that this is false. Then by part one, $f(S) = f^{p',q'}(S)$, where either $p' < \dot{p}$ or $q' < \dot{q}$. Without loss of generality, assume that $p' < \dot{p}$. Let $x \in f(S)$ such that $x < \inf f^{\dot{p},\dot{q}}(S)$.

Let π be a permutation of N such that, for all $i, j \in N$, $\inf S_i < \inf S_j$ implies that $\pi(i) < \pi(j)$. Observe that $\inf f^{1,1}(S) \leq \inf S_{\pi^{-1}(p')} < x < \inf S_{\pi^{-1}(p*)} < \sup f^{1,1}(S)$. Let $\phi \in \Phi^+$ such that (a) $\phi \inf f^{1,1}(S) > \dot{p}-1$, (b) $\phi \inf S_{\pi^{-1}(p')} = \dot{p} - \frac{1}{2}$, (c) $\phi x = \dot{p} - \frac{1}{4}$, (d) $\phi \inf S_{\pi^{-1}(p'*} > n$, and (e) $\phi \sup f^{1,1}(S) < n+1$.

Note that $\phi S_i \subseteq \pi S^1_i$ for all $i \in N$. To see that this is true, observe that for j such that $\pi(j) < \dot{p}$, $\inf(\phi S_j) > \dot{p} - 1 \geq \pi(j) = \inf S^1_{\pi(j)}$ and $\sup(\phi S_j) < n+1 \leq n + \pi(j) = \sup S^1_{\pi(j)}$, and for j such that $\pi(j) \geq \dot{p}$, $\inf(\phi S_j) > n \geq \pi(j) = \inf S^1_{\pi(j)}$ and $\sup(\phi S_j) < n+1 \leq n+\pi(j) = \sup S^1_{\pi(j)}$.

Because f satisfies responsiveness and anonymity, $\phi f(S) \subseteq f(S^1)$. Because $\inf f^{\dot{p},\dot{q}}(S) \in f(S)$, $\phi \inf f^{\dot{p},\dot{q}}(S) \in \phi f(S)$. By construction, $\phi \inf f^{\dot{p},\dot{q}}(S) = \dot{p}$. But this implies that $\dot{p} \in f(S^1)$, a contradiction.

Next, I show that $f^{\dot{p},\dot{q}}(S) \subseteq f(S)$. Let $x \in f^{\dot{p},\dot{q}}(S)$. I show that $x \in f(S)$.

For $i \in N$, choose $x_i \in \mathbb{R}$ such that (a) $x_i \in S_i$, (b) $x_i \neq x_j$ for $j \neq i$, (c) $|\{i \in N : x_i \leq x\}| = \dot{p}$, and (d) $|\{i \in N : x_i \geq x\}| = \dot{q}$. Let $\varepsilon > 0$ such that (i) for all $i \in N$, $(x_i - \varepsilon, x_i + \varepsilon) \in S_i)$, and (ii) $\varepsilon < \min_{i,j} |x_i - x_j|$. Define $X \in \Sigma^N$ such that $X \equiv (x_i - \varepsilon, x_i + \varepsilon)$.

Let π' be a permutation of N such that, for all $i, j \in N$, $x_i < x_j$ implies that $\pi(i) < \pi(j)$. Let $\phi \in \Phi^+$ such that for all $i \in N$, $\phi(x_i - \varepsilon) = 2\pi(i) - 1$ and $\phi(x_i + \varepsilon) = 2\pi(i)$. Note that $\phi(x) > 2\dot{p} - 1$ and that $\phi(x) < 2(n+1-\dot{q})$.

Note that $\pi\phi X = S^n$, which implies that $f(\pi\phi X) = f(S^n) = (2\dot{p}-1, 2(n+1-\dot{q}))$. By neutrality and anonymity, it follows that $\phi f(X) = (2\dot{p}-1, 2(n+1-\dot{q}))$ which implies that $f(X) = (x_{\dot{p}} - \varepsilon, x_{n+1-\dot{q}} + \varepsilon)$, and hence, $x \in f(X)$. Because $X_i \subseteq S_i$ for all $i \in N$, it follows that $x \in f(S)$.

Part Four: The last step is to show that $\dot{p} + \dot{q} \leq n + 1$. Suppose contrariwise that $\dot{p} + \dot{q} > n + 1$. Then $\dot{p} - 1 \geq n + 1 - \dot{q}$. Thus $\inf f(S^n) = \inf S^n_{\dot{p}} = 2\dot{p} - 1$ and $\sup f(S^n)) = \sup S^n_{n+1-\dot{q}} = 2(n + 1 - \dot{q})$. Because $\dot{p} - 1 \geq n + 1 - \dot{q}$ it follows that $2\dot{p} - 1 > 2(\dot{p} - 1) \geq 2(n + 1 - \dot{q})$. This implies that $\inf f(S^n) > \sup f(S^n)$, a contradiction. □

Proof of Theorem 2. That symmetric endpoint rules satisfy the axioms is trivial. Let f satisfy the axioms. Because strong neutrality implies weak neutrality, f is an endpoint rule with quotas p and q. I show that $p = q$. It is sufficient to show that for all $S \in \Sigma^N$, $f^{p,q}(S) = f^{q,p}(S)$.

Let $S \in \Sigma^N$ and let $\phi \in \Phi$ be the transformation such that $\phi(x) = -x$ for all $x \in \mathbb{R}$. Note that $f^{p,q}(S) = f^{p,q}(\phi\phi S)$ and, by strong neutrality, that $f^{p,q}(\phi\phi S) = \phi f^{p,q}(\phi S)$. It remains to be shown that $f^{p,q}(\phi(S)) = \phi(f^{q,p}(S))$. To see this, note that $\phi(f^{p,q}(S)) = \{\phi(x) : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}$. Because $\phi = \phi^{-1}$, it follows that $\phi(f^{p,q}(S)) = \{x : |G^+(S, \phi(x))| \geq p \text{ and } |G^-(S, \phi(x))| \geq q\}$. Because ϕ is decreasing, $G^+(S, \phi(x)) = G^-(\phi(S), x)$ and $G^-(S, \phi(x)) = G^+(\phi(S), x)$. Hence, $\phi(f^{p,q}(S)) = \{x : |G^-(\phi(S), x)| \geq p \text{ and } |G^+(\phi(S), x)| \geq q\} = f^{q,p}(\phi(S))$. □

For two points $x, y \in \mathbb{R}$, let $B(x, y) = \{z \in \mathbb{R} : x \leq z \leq y \text{ or } y \leq z \leq x\}$.

Lower property: For $i \in N$ and $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$, either (a) $\inf f(S) = \inf f(T)$ or (b) $\inf f(S) \in B(\inf S_i, \inf f(T))$ and $\inf f(T) \in B(\inf T_i, \inf f(S))$.

Upper property: For $i \in N$ and $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$, either (a) $\sup f(S) = \sup f(T)$ or (b) $\sup f(S) \in B(\sup S_i, \sup f(T))$ and $\sup f(T) \in B(\sup T_i, \sup f(S))$.

I next state and prove the following lemma.

Lemma 2. *Strategyproofness implies both the lower property and the upper property.*

Proof. Let $i \in N$ and let $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$. Let f satisfy strategyproofness.

Strategyproofness implies that for every preference $\succeq^S \in \mathcal{P}$ with peak S_i, $f(S) \succeq^S f(T)$. This implies that (i) $f(S) \in \mathcal{B}(S_i, f(T))$. A similar argument shows that (ii) $f(T) \in \mathcal{B}(T_i, f(S))$. To prove the lower property, there are three cases:

Case 1: $\inf f(S) \in B(\inf S_i, \inf T_i)$. Statement (ii) implies that $\inf f(T) \in B(\inf T_i, \inf f(S))$, which implies that $\inf f(S) \in B(\inf S_i, \inf f(T))$.

Case 2. $\inf f(S) > \inf S_i, \inf T_i$. By statement (i) $\inf f(S) \leq \inf f(T)$. By statement (ii) $\inf f(T) \leq \inf f(S)$. This implies that $\inf f(S) = \inf f(T)$.

Case 3. $\inf f(S) < \inf S_i, \inf T_i$. By statement (i) $\inf f(T) \leq \inf f(S)$. By statement (ii) $\inf f(S) \leq \inf f(T)$. This implies that $\inf f(S) = \inf f(T)$.

The upper property is proven in a similar fashion. □

Proof of Proposition 1: I will show that strategyproofness implies the independent aggregation of the lower endpoints. That strategyproofness implies the independent aggregation of the upper endpoints follows from a dual argument.

Let $S, T \in \Sigma^N$ such that $\inf S_i = \inf T_i$ for all $i \in N$ and let f satisfy strategyproofness. I will show that $\inf f(S) = \inf f(T)$. For $j \in N$ let $S^j \in \Sigma^N$ such that $S_i^j = S_i$ for $i \leq j$ and such that $S_i^j = T_i$ otherwise. Define $S^0 \equiv S$.

Let $k \in N$. It is sufficient to prove that $\inf f(S^{k-1}) = \inf f(S^k)$.

Because f satisfies strategyproofness it follows from Lemma 2 that f satisfies the lower property. By the lower property, either (a) $\inf f(S^{k-1}) = \inf f(S^k)$ or (b) $\inf f(S^{k-1}) \in B(\inf S_k^{k-1}, \inf f(S^k)$ and $\inf f(S^k) \in B(\inf S_k^k, \inf f(S^{k-1})$.

Because $\inf f(S^{k-1}) \in B(\inf S_k^{k-1}, \inf f(S^k))$ it follows that either (i) $\inf S_k^{k-1} \geq \inf f(S^{k-1}) \geq \inf f(S^k)$ or (ii) $\inf S_k^{k-1} \leq \inf f(S^{k-1}) \leq \inf f(S^k)$. That $\inf f(S^k) \in B(\inf S_k^k, \inf f(S^{k-1}))$ implies that either (iii) $\inf S_k^k \geq \inf f(S^k) \geq \inf f(S^{k-1})$ or (iv) $\inf S_k^k \leq \inf f(S^k) \leq \inf f(S^{k-1})$. The combinations of (i) and (iii) and of (ii) and (iv) directly imply that $\inf f(S^{k-1}) = \inf f(S^k)$. The combinations of (i) and (iv) and of (ii) and (iii), combined with the fact that $\inf S_k^{k-1} = \inf S_k^k$, imply that $\inf f(S^{k-1}) = \inf f(S^k)$. □

Proof of Claim 1. That MED satisfies anonymity and strategyproofness is straightforward. Let f satisfy anonymity and strategyproofness. I show there exists $P \in \overline{\Sigma}^{n+1}$ such that $f(S) = \text{MED}(S, P)$ for all $S \in \Sigma^N$.

For a function $g : \mathbb{R}^N \to \mathbb{R}$, define g to be anonymous if for every permutation π of N, $g(\mathbf{x}_1, \ldots, \mathbf{x}_n) = g(\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(n)})$. Define g to be strategyproof if (a) for every agent i with single-peaked preferences $\succeq_i$ over $\mathbb{R}$ and associated peak p_i, and (b) for every $\mathbf{x} \in \mathbb{R}^N$, $g(p_i, \mathbf{x}_{-i}) \succeq_i g(\mathbf{x})$.

By Proposition 1, because f is strategyproof it satisfies independent aggregation of endpoints. Consequently, there exists $\underline{g}, \overline{g} : \mathbb{R}^N \to \mathbb{R}$ such that for all $S \in \Sigma^N$, $f(S) = (\underline{g}(\{\inf S_i\}), \overline{g}(\{\sup S_i\}))$. I first prove that $\underline{g}$ is anonymous and strategyproof. A similar argument shows that $\overline{g}$ is anonymous and strategyproof.

To show that $\underline{g}$ is anonymous, let π be a permutation of N and let $S \in \Sigma^N$. Then $f(\pi S) = \underline{g}(\{\inf S_{\pi(i)}\})$. By anonymity, $f(S) = f(\pi S)$; therefore, $\underline{g}(\{\inf S_i\}) = \underline{g}(\{\inf S_{\pi(i)}\})$.

To show that g is strategyproof, let $S \in \Sigma^N$, $i \in N$, and $\succeq_i \in \mathcal{P}$. Let $T \in \Sigma^N$ such that $T_i = S_i^*$ (agent i's ideal point) and such that $T_j = S_j$ for all $j \neq i$. Because strategyproofness implies independent aggregation of endpoints, there is a single-peaked preference relation $\succeq_i^-$ over $\mathbb{R}$ with associated peak $p_i = \inf S_i^*$. Define $\mathbf{x} = (\inf S_1, \cdots, \inf S_n)$. Because f is strategyproof, it satisfies the lower property. Therefore either (i) $\inf f(S) = \inf f(T)$ or (ii) $\inf f(S) \in B(\inf S_i, \inf f(T))$ and $\inf f(T) \in B(\inf T_i, \inf f(S))$. If (i) then $g(p_i, \mathbf{x}_{-i}) = \inf f(S) = \inf f(T) = g(\mathbf{x})$, which implies that $g(p_i, \mathbf{x}_{-i}) \succeq_i g(\mathbf{x})$, If (ii) then $g(p_i, \mathbf{x}_{-i}) \in B(p_i, g(\mathbf{x}))$, which implies that $g(p_i, \mathbf{x}_{-i}) \succeq_i g(\mathbf{x})$. Thus $\underline{g}$ is strategyproof.

Because $\underline{g}$ is anonymous and strategyproof, it follows from Moulin (1980, Proposition 2) that there exist $n+1$ real numbers $\alpha_1, \ldots, \alpha_{n+1} \in \overline{\mathbb{R}}$ such that, for all $x \in \mathbb{R}^N$, $\underline{g}(x) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1})$, where m is the function that selects the median element of $\{x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1}\}$. Similarly, we can show that there exist $n+1$ real numbers $\beta_1, \ldots, \beta_{n+1} \in \overline{\mathbb{R}}$ such that, for all $x \in \mathbb{R}^N$, $\overline{g}(x) = m(x_1, \ldots, x_n, \beta_1, \ldots, \beta_{n+1})$.

To finish the proof, it is sufficient to show that there is a permutation π over $\{1, \ldots, n+1\}$ such that $\alpha_i < \beta_{\pi(i)}$ for all $i \in \{1, \ldots, n+1\}$. Suppose by means of contradiction that no such permutation π exists. Then we can order the αs so that $\alpha_{(1)} < \cdots < \alpha_{(n+1)}$, and let $k \leq n+1$ such that $|\{i : \beta_i \leq \alpha_{(k)}\}| \geq k$. Let $c = n+1-k$. Let $S \in \Sigma^N$ such that for $i = 1, \ldots, c$, $\inf S_i > \alpha_{(k)}$, and for $i = c+1, \ldots, n+1$, $\sup S_i < \alpha(k)$. Then $\inf f(S) = \underline{g}(\{infS_i\}) = m(\{\inf S_i\}, \alpha_{(1)}, \ldots, \alpha_{(n+1)}) = \alpha_{(k)} > m(\{\sup S_i\}, \beta_{(1)}, \ldots, \beta_{(n+1)}) = \sup f(S)$, a contradiction that proves the claim. □

Proof of Theorem 3. That endpoint rules satisfy the axioms follows from Theorem 1, Claim 1, and the fact that neutrality implies translation equivariance.

Let f satisfy the three axioms. That f satisfies anonymity and strategyproofness implies, by Claim 1, that there exists $P \in \overline{\Sigma}^{n+1}$ such that $f(S) = \text{MED}(S, P)$ for all $S \in \Sigma^N$. Let $P \in \overline{\Sigma}^{n+1}$ such that $f(S) = \text{MED}(S, P)$ for all $S \in \Sigma^N$. Let $y, z \in \mathbb{R}$ such, for all $x \in \mathbb{R} \cap (\cup_{i \leq n+1} \{\inf P_i, \sup P_i\})$, $y \leq x \leq z$. Let $u = |\{P_i : P+i = (-\infty, -\infty)\}|$, let $v = |\{P_i : P+i = (-\infty, \infty)\}|$, and let let $w = |\{P_i : P+i = (\infty, \infty)\}|$. Note that $u, w \geq 1$, otherwise there is no f for which such a P exists. I show that $u+v+w = n+1$.

Let $S \in \Sigma^N$ such that for all $i, j \in N$, $i \neq j$, $\inf S_i \neq \inf S_j$ and $\sup S_i \neq \sup S_j$.

Let $c = z + 1 - \inf_i \in N \inf S_i$. Then for all S_i, $\inf S_i + c > z$. This implies that $f([S+c]) = \text{MED}(S, P) = f^{w,n+1-w-v}([S+c])$. By translation equivariance $f([S+c]) = [f(S)+c]$ Also, by translation equivariance, $f^{w,n+1-w-v}([S+c]) = [f^{w,n+1-w-v}(S)+c]$. Together this implies that $f(S) = f^{w,n+1-w-v}(S)$.

Next, let $d = y + 1 - \sup_i \in N \sup S_i$. Then for all S_i, $\sup S_i + d < y$. This implies that $f([S+d]) = \text{MED}(S, P) = f^{n+1-u-v,u}([S+d])$. By translation equivariance $f([S+d]) = [f(S)+d]$ Also, by translation equivariance, $f^{n+1-u-v,u}([S+d]) = [f^{n+1-u-v,u}(S)+d]$. Together this implies that $f(S) = f^{n+1-u-v,u}(S)$.

It follows that $f(S) = f^{w,n+1-w-v}(S) = f^{n+1-u-v,u}(S)$, Thus $w = n+1-u-v$, and therefore $n+1 = u+v+w$. □

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Appendix for Referees (not for publication)

The "If" part of of Theorem 1. Let $f^{p,q}$ be an endpoint rule. I will show that $f^{p,q}$ satisfies the four axioms.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$, and let $x \in f^{p,q}(S)$. I will show that $x \in f^{p,q}(T)$. Because $x \in f^{p,q}(S)$ it follows that $|G^+(S,x)| \geq p$ and $|G^-(S,x)| \geq q$. Because $S_i \subseteq T_i$ for all $i \in N$ it follows that $G^+(S,x) \subseteq G^+(T,x)$ and $G^-(S,x) \subseteq G^-(T,x)$. This implies that $|G^+(T,x)| \geq p$ and $|G^-(T,x)| \geq q$ and therefore that $x \in f^{p,q}(T)$.

Anonymity: Letπ be a permutation of N, let $S \in \Sigma^N$, and let $x \in f^{p,q}(S)$. Because $x \in f^{p,q}(S)$ it follows that $p \leq |G^+(S,x)| = |G^+(\pi S,x)| \geq p$ and $q \leq |G^-(S,x)| = |G^-(\pi S,x)| \geq q$. Hence $x \in f^{p,q}(\pi S)$. Because the choice of π is arbitrary, it follows that $f^{p,q}(S) = f^{p,q}(\pi S)$ for all $S \in \Sigma^N$.

Continuity: Let $\delta > 0$, let $\varepsilon = \frac{\delta}{3}$, and let $S \in \Sigma^N$. Let $S' \in \Sigma^N$ such that $d(S_i, S_i') < \varepsilon$ for all $i \in N$. It follows that $|\inf f^{p,q}(S) - \inf f^{p,q}(S')| < \varepsilon$ and $|\sup f^{p,q}(S) - \sup f^{p,q}(S')| < \varepsilon$. As a consequence, $d(f^{p,q}(S), f^{p,q}(S')) < 2\varepsilon < \delta$.

Weak Neutrality: Let $\phi \in \Phi^+$ and $S \in \Sigma^N$. Let $x \in \phi(f(S))$. This implies that $\phi^{-1}(x) \in f(S)$ and therefore that $|G^+(S,\phi^{-1}(x))| \geq p$ and $|G^-(S,\phi^{-1}(x))| \geq q$. Because ϕ is increasing, $G^+(S,\phi^{-1}(x)) = G^+(\phi(S),\phi(\phi^{-1}(x))) = G^+(\phi(S),x)$ and $G^-(S,\phi^{-1}(x)) = G^-(\phi(S),x)$. It follows that $|G^+(\phi(S),x)| \geq p$ and $|G^-(\phi(S),x)| \geq q$ and therefore that $x \in f^{p,q}(\phi S)$. Because the choice of ϕ is arbitrary, it follows that $f^{p,q}(S) = f^{p,q}(\pi S)$ for all $S \in \Sigma^N$. □

The "If" part of of Theorem 2. Let $f^{p,p}$ be a symmetric endpoint rule. By Theorem 1, because $f^{p,p}$ is an endpoint rule it satisfies responsiveness, anonymity, and continuity.

To show that $f^{p,p}$ satisfies strong neutrality, let $\phi \in \Phi$ and $S \in \Sigma^N$. Let $x \in \phi(f(S))$. This implies that $\phi^{-1}(x) \in f(S)$, and therefore that $|G^+(S,\phi^{-1}(x))| \geq p$ and $|G^-(S,\phi^{-1}(x))| \geq p$. If ϕ is increasing, then $G^+(S,\phi^{-1}(x)) = G^+(\phi(S),\phi(\phi^{-1}(x))) = G^+(\phi(S),x)$ and $G^-(S,\phi^{-1}(x)) = G^-(\phi(S),x)$; if ϕ is decreasing, $G^+(S,\phi^{-1}(x)) = G^-(\phi(S),x)$ and $G^-(S,\phi^{-1}(x)) = G^+(\phi(S),x)$. Hence, because $|G^+(S,\phi^{-1}(x))| \geq p$ and $|G^-(S,\phi^{-1}(x))| \geq p$, it follows that $|G^+(\phi(S),x)| \geq p$ and $|G^-(\phi(S),x)| \geq p$, and therefore that $x \in f^{p,p}(\phi(S))$. Because the choice of ϕ is arbitrary, it follows that $f^{p,p}(S) = f^{p,p}(\phi(S))$ for all $S \in \Sigma^N$. □

The "If" part of Claim 1. The rule MED trivially satisfies anonymity. I show that it satisfies strategyproofness.

First, I show that the MED satisfies independent aggregation of lower endpoints. A dual argument shows that MED satisfies independent aggregation of upper endpoints.

To see this, let $S, S' \in \Sigma^N$ such that $\inf S_i = \inf S_i'$ for all i. For $T \in \Sigma^N$, $\text{MED}(T,P) = \{x \in \mathbb{R} : \min\{|\{i : (-\infty, x] \cap P_i \neq \varnothing\}| + |\{i : (-\infty, x] \cap T_i \neq \varnothing\}|, |\{i : [x, +\infty) \cap P_i \neq \varnothing\}| + |\{i : [x, +\infty) \cap T_i \neq \varnothing\}|\} \geq n+1\}$. This implies that for all $\inf \text{MED}(T,P) = \min\{x : |\{x \geq \inf P_i\}| + |\{x \geq \inf T_i\}| \geq n+1\}$. Because P

is fixed and because $\inf S_i = \inf S'_i$ for all $i \in N$, it follows that $\inf \text{MED}(S, P) = \inf \text{MED}(S', P)$.

Let $P \in \overline{\Sigma}^{N+1}$, $S \in \Sigma^N$, $i \in N$, and and $\succeq_i \in \mathcal{P}$. I show that the aggregation of the lower endpoints is strategyproof; that is, either $\inf f(S) \leq \inf f(S_1, ..., S_i^*, ..., S_n) \leq \inf S_i^*$ or that $\inf f(S) \geq \inf f(S_1, ..., S_i^*, ..., S_n) \geq \inf S_i^*$. That the aggregation of the upper endpoints is strategyproof follows from a dual argument.

Let $\mathbf{x} \in \mathbb{R}^{2N+1}$ such that, for $i \in N$, $\mathbf{x}_i = \inf S_i$ and, for $i \in \{N+1, \cdots, 2N+1\}$, $\mathbf{x}_i = \inf P_i$. Let $p^* = \inf S_i^*$. Because MED satisfies independent aggregation of lower endpoints, there is a function $g : \mathbb{R}^{2N+1} \to \mathbb{R}$ such that $g(\mathbf{x}) = \min\{x \in \mathbb{R} : |\{j \in \{1, \cdots, 2N+1\} : x \geq \mathbf{x}_j\}| \geq n+1\} = \max\{x \in \mathbb{R} : |\{j \in \{1, \cdots, 2N+1\} : x \leq \mathbf{x}_j\}| \geq n+1\}$.

If $g(p^*, \mathbf{x}_{-i}) < p^*$, we are done.

Suppose that $g(p^*, \mathbf{x}_{-i}) < p^*$. Then $|\{j \in \{1, \cdots, 2N+1\} \setminus i : g(p^*, \mathbf{x}_{-i}) \geq \mathbf{x}_j\}| \geq n+1$. This implies that $|\{j \in \{1, \cdots, 2N+1\} : g(p^*, \mathbf{x}_{-i}) \geq \mathbf{x}_j\}| \geq n+1$. Consequently, $g(\mathbf{x}) \leq g(p^*, \mathbf{x}_{-i}) < p^*$, and therefore $\inf f(S) \leq \inf f(S_1, ..., S_i^*, ..., S_n) \leq \inf S_i^*$.

Next, suppose that $g(p^*, \mathbf{x}_{-i}) > p^*$. Then $|\{j \in \{1, \cdots, 2N+1\} \setminus i : g(p^*, \mathbf{x}_{-i}) \leq \mathbf{x}_j\}| \geq n+1$. This implies that $|\{j \in \{1, \cdots, 2N+1\} : g(p^*, \mathbf{x}_{-i}) \leq \mathbf{x}_j\}| \geq n+1$. Consequently, $g(\mathbf{x}) \geq g(p^*, \mathbf{x}_{-i}) < p^*$, and therefore $\inf f(S) \geq \inf f(S_1, ..., S_i^*, ..., S_n) > \inf S_i^*$. □

Independence of the Axioms. To simplify these results I begin by showing that strategyproofness implies responsiveness and continuity.

Proof. Let f satisfies strategyproofness. It follows from Lemma 2 that f satisfies the lower and upper properties.

Part One. I show that strategyproofness implies responsiveness.

Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. For $j \in N$ let $S^j \in \Sigma^N$ such that $S_i^j = S_i$ for $i \leq j$ and such that $S_i^j = T_i$ otherwise. Define $S^0 \equiv S$.

Let $k \in N$. It is sufficient to show that $f(S^{k-1}) \subseteq f(S^k)$.

Because $S_k^{k-1} \subseteq S_k^k$, it follows that $\inf S_k^k \leq \inf S_k^{k-1}$. For all $i \neq k$, $S_i^{k-1} = S_i^k$. Therefore, by the lower property, either $\inf f(S^k) = \inf f(S^{k-1})$ or else $\inf S_k^k \leq \inf f(S^k) \leq \inf f(S^{k-1}) \leq \inf S_k^{k-1}$. It follows that $\inf f(S^k) \leq \inf f(S^{k-1})$.

Also because $S_k^{k-1} \subseteq S_k^k$, it follows that $\sup S_k^{k-1} \leq \sup S_k^k$. Therefore, by the upper property, either $\sup f(S^k) = \sup f(S^{k-1})$ or else $\sup S_k^{k-1} \leq \sup f(S^{k-1}) \leq \sup f(S^k) \leq \sup S_k^k$. It follows that $\sup f(S^{k-1}) \leq \sup f(S^k)$.

From the fact that $\inf f(S^k) \leq \inf f(S^{k-1}) \leq \sup f(S^{k-1}) \leq \sup f(S^k)$ it follows that $f(S^{k-1}) \subseteq f(S^k)$.

Part Two. I show that strategyproofness implies continuity.

Let f satisfy strategyproofness. Suppose, by way of contradiction, that f does not satisfy continuity. Then there exists Y in τ such that $f^{-1}(Y) \notin \tau^N$. Because $f^{-1}(Y) \notin \tau^N$, there exists $S \in f^{-1}(Y)$ and $i \in N$ such that $\inf T_i < \inf S_i < \sup S_i <$

$\sup T_i$ implies that $T \notin f^{-1}(Y)$. Because $f(S) \in Y$, it follows that there exists $W \in f^{-1}(Y)$ such that $\inf f(W) < \inf f(S) < \sup f(S) < \sup f(W)$.

Let $T \in \Sigma^N$ such that $\inf T_i < \inf S_i < \sup S_i < \sup T_i$, $\inf S_i - \inf T_i < \inf f(S) - \inf f(W)$, $\sup T_i - \sup S_i < \sup f(W) - \sup f(S)$, and $T_j = S_j$ for all $j \neq i$. By construction, $f(T) \notin Y$. By responsiveness, $f(S) \in f(T)$.

Together, this implies that $\inf f(T) < \inf f(W) < \inf f(S)$, and therefore that $\inf f(S) - \inf f(T) > \inf f(S) - \inf f(W) > \inf S_i - \inf T_i$.

By strategyproofness, $\inf S_i > \inf T_i$ and $\inf f(S) > \inf f(T)$ imply that $\inf T_i \leq \inf f(T) < \inf f(S) \leq \inf S_i$. Therefore $\inf S_i - \inf T_i \geq \inf f(S) - \inf f(T)$, a contradiction. □

The independence of the axioms in Theorems 1 and 2 follows from rules 1–4. The independence of the axioms in Claim 1 and Theorem 3 follows from rules 1–3.

Rule 1: This rule satisfies responsiveness, anonymity, weak neutrality, strong neutrality, and translation equivariance, but fails continuity and therefore strategyproofness.

Let S^n be the profile such that $S^n_i = (2i-1, 2i)$ for all i. $f(S) = f^{2,2}(S)$ for all S such that $\pi S = \phi S^n$ for some π and ϕ, otherwise $f(S) = f^{1,1}(S)$. (For $n \geq 3$.)

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. If there exists π and ϕ such that $\pi S = \phi S^n$, then $f(S) = f^{2,2}(S) \subseteq f^{2,2}(T) \subseteq f^{1,1}(T)$. It follows that $f(S) \subseteq f(T)$.

Anonymity: Let $S \in \Sigma^N$ and let π' be a permutation of N. If there exists π and ϕ such that $\pi S = \phi S^n$, then $\pi^*(\pi' S) = \phi S^n$ for $\pi^* = \pi\pi'^{-1}$. In this case, $f(\pi' S) = f^{2,2}(\pi' S) = f^{2,2}(S) = f(S)$. If there does not exist π and ϕ such that $\pi S = \phi S^n$, then there does not exist π and ϕ such that $\pi(\pi' S) \neq \phi S^n$. In this case, $f(\pi' S) = f^{1,1}(\pi' S) = f^{1,1}(S) = f(S)$. It follows that for all S, $f(\pi' S) = f(S)$.

Strong neutrality: Let $S \in \Sigma^N$ and let $\phi' \in \Phi$. If there exists π and ϕ such that $\pi S = \phi S^n$, then $\pi(\phi' S) = \phi^* S^n$ for $\phi^* = \phi'\phi$. In this case, $f(\phi' S) = f^{2,2}(\phi' S) = \phi' f^{2,2}(S) = \phi' f(S)$. If there does not exist π and ϕ such that $\pi S = \phi S^n$, then there does not exists π and ϕ such that $\pi(\phi' S) \neq \phi S^n$. In this case, $f(\phi' S) = f^{1,1}(\phi' S) = \phi' f^{1,1}(S) = \phi' f(S)$. It follows that for all S, $f(\phi' S) = f(S)$.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Translation equivariance: Any rule that satisfies strong neutrality necessarily satisfies translation equivariance.

Continuity: Let $n = 3$. Let $Y \subset \Sigma$ such that $Y = \{(a, b) \in \Sigma : a > 2$ and $b < 6$. Note that $Y \in \tau$. Let $\varepsilon > 0$ and let $S, T \in \Sigma^N$ such that $S_1 = T_1 = (0, 2)$, $S_2 = T_2 = (3, 5 - 0.5\varepsilon)$, $S_3 = (5 - \varepsilon, 7)$, and $T_3 = (5, 7)$. By construction, there exists π and ϕ such that $\pi T = \phi T^n$. Consequently, $f(T) = f^{2,2}(S) = (3, 5 - 0.5\varepsilon) \in Y$. Therefore $T \in f^{-1}(Y) \in \tau^N$. Because $f^{-1}(Y) \in \tau^N$ it follows that for ε small enough, $S \in f^{-1}(Y) \in \tau^N$. But as there does not exist π and ϕ such that $\pi S = \phi S^n$, it follows that $f(S) = f^{1,1}(S) = (0, 1) \notin Y$, a contradiction.

Rule 2: This rule satisfies anonymity, responsiveness, continuity, and strategyproofness, but fails translation equivariance, and therefore weak neutrality and strong neutrality.

$f(S) = (1, 2)$ for all $S \in \Sigma^N$.

Anonymity: Let $S \in \Sigma^N$ and let π be a permutation of N. Then $f(S) = (1, 2)$ and $f(\pi S) = (1, 2) = f(S)$.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. Then $f(T) = (1, 2)$ and $f(S) = (1, 2) \subseteq f(T)$.

Continuity: Let $Y \in \tau$. If $(1, 2) \in Y$ then $f^{-1}(Y) = \Sigma^N \in \tau^N$. If $(1, 2) \notin Y$ then $f^{-1}(Y) = \varnothing \in \tau^N$.

Strategyproofness: The outcome is variant to changes in the intervals and is therefore strategyproof.

Translation equivariance: Let $S \in \Sigma^N$ such that $b = x+1$. Then $f([S+b]) = (1, 2)$ but $[f(S) + b] = (2, 3)$, a contradiction.

Weak neutrality: Any rule that fails translation equivariance necessarily fails weak neutrality.

Strong neutrality: Any rule that fails translation equivariance necessarily fails strong neutrality.

Rule 3: This rule satisfies responsiveness, continuity, translation equivariance, weak neutrality, strong neutrality, and strategyproofness, but fails anonymity.

$f(S) = S_1$ for all $S \in \Sigma^N$.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. Then $S_1 \subseteq T_1$ which implies that $f(S) \subseteq f(T)$.

Continuity: Let $Y \in \tau$. Then $f^{-1}(Y) = Y \times \Sigma^{N \setminus \{1\}} \in \tau^N$.

Strong neutrality: Let $S \in \Sigma^N$ and let $\phi \in \Phi$. Then $f(\phi S) = \phi S_1$ and $\phi f(S) = \phi S_1$, which implies that $f(\phi S) = \phi f(S)$.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies translation equivariance.

Anonymity: Let $S \in \Sigma^N$ such that $S_1 \neq S_2$ and let π be a permutation such that $\pi(1) = 2$. Then $f(S) = S_1$ but $f(\pi S) = S_2 \neq S_1$.

Strategyproofness. Agent 1 will reveal her preferred interval, and the rule is invariant to the other agents.

Rule 4: This rule satisfies anonymity, continuity, translation equivariance, weak neutrality, and strong neutrality, but fails responsiveness and therefore strategyproofness.

$f(S) = (\inf f^{1,n}(S), \max\{\sup f^{1,n}(S), \inf S_1, \ldots, \inf S_n\})$ for all $S \in \Sigma^N$.

Anonymity: Let $S \in \Sigma^N$ and let π be a permutation of N. Then $f(\pi S) = (\inf f^{1,n}(\pi S), \max\{\sup f^{1,n}(\pi S), \inf \pi S_1, \ldots, \inf \pi S_n\})$. Note that

$\inf f^{1,n}(\pi S) = \inf f^{1,n}(S)$, $\sup f^{1,n}(\pi S) = \sup f^{1,n}(S)$, and $\{\inf \pi S_1, \ldots, \inf \pi S_n\} = \{\inf S_1, \ldots, \inf S_n\}$, which implies that $f(\pi S) = f(S)$.

Continuity: Let $S \in \Sigma^N$. Let $Y \in \tau$ such that $f(S) \in Y$. I show that there exists $U \in \tau^N$ such that $f(U) \subseteq Y$. Because $f(S) \in Y \in \tau$ it follows that there exists $\varepsilon > 0$ such that $(\inf f(S) - \varepsilon, \sup f(S) + \varepsilon) \in Y$. Let $U = \{T \in \Sigma^N : |\inf S_i - \inf T_i|, |\sup S_i - \sup T_i| < \varepsilon\}$. By construction, $S \in U$. Note that $U \in \tau^N$. Let $T \in U$. Then $\inf f(T) > \inf f(S) - \varepsilon$ and $\sup f(T) < \sup f(S) + \varepsilon$ which implies that $f(T) \in Y'$ and consequently that $f(U) \subseteq Y$.

Strong neutrality: Let $S \in \Sigma^N$ and let $\phi \in \Phi$.
Then $f(\phi S) = (\inf f^{1,n}(\phi S), \max\{\sup f^{1,n}(\phi S), \inf(\phi S_1), \ldots, \inf(\phi S_n)\}) =$
$(\phi \inf f^{1,n}(S), \max\{\phi \sup f^{1,n}(S), \phi \inf S_1, \ldots, \phi \inf S_n\}) =$
$\phi(\inf f^{1,n}(S), \max\{\sup f^{1,n}(S), \inf S_1, \ldots, \inf S_n\}) = \phi f(S)$,

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies translation equivariance.

Responsiveness: Let $S \in \Sigma^N$ such that $S_i = (0, 2i)$ for all $i \in N$. Then $S_i^n \subseteq S_i$ for all $i \in N$. However, $f(S^n) = (0, 2i - 1)$ while $f(S) = (0, 2)$, contradicting the fact that $f(S^n) \subseteq f(S)$. □