Interval Aggregation

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Abstract

I propose a model of aggregation of intervals relevant in two contexts: the study of legislative delegation and the study of legal standards of behavior. Six axioms: responsiveness, anonymity, continuity, strategyproofness, and two variants of neutrality are then used to prove several important results about a new class of aggregation methods called endpoint rules. The class of endpoint rules ranges from extreme tolerance at one end (allowing anything permitted by anyone) to a form of majoritarianism (the median rule), at the other.

1 Introduction

Social institutions often give agents limited freedom to make decisions. This freedom of choice is analogous to the delegation of discretion long studied in the principal-agent literature (see Holmström, 1977, 1984), but with one important difference: in the case of social institutions, the principal is often a group of individuals. These individuals may disagree about the extent of, or the limits on, the agents' discretion. This paper provides a theory for how to resolve this disagreement.

This problem arises in (at least) two important contexts. First, legislatures delegate a wide berth of discretion to enable administrative agencies to make decisions consistent with their expertise.[1] Second, courts permit agents significant latitude to

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[1] This delegation of discretion is considered to be so important that Federal law prohibits courts from holding the U.S. Government liable for torts "based upon the exercise or performance or the failure to exercise or perform a discretionary function or duty. . . , whether or not the discretion involved be abused." 28 U.S.C. 2680(a).

behave as they wish, provided their behavior comports with a legal standard of behavior. For example, the common law of negligence restricts the imposition of liability for accidents unless the injurer has failed to act as would a reasonable person under the circumstances (see Miller and Perry, 2012).

The legislative problem has been studied extensively by political scientists interested in the problem of bureaucratic drift, or how administrative agency decisions can deviate away from those preferred by the legislature (see Epstein and O'Halloran, 1994, 1996, 1999, 2008; Huber and Shipan, 2002, 2006; Alonso and Matouschek, 2008; Gailmard, 2009; Gailmard and Patty, 2012; Callander and Krehbiel, 2014). This literature, however, does not go far to answer the question of how to combine the judgments of the many legislators. Rather, these papers tend to assume away the difficulty of combining legislators' preferences by treating the legislature as a unitary entity; for example, by identifying the legislature with a median legislator (see, for example Epstein and O'Halloran, 1994).

This paper neither assumes the existence of a well-defined median legislator nor attempts to solve the long standing problem of aggregating preferences identified by Arrow (1963). Rather than combine the legislators' preferences, I instead combine their judgments—the delegation decision that would be made by each legislator, were that legislator's preferences to fully determine the outcome. In this literature, these judgments are generally modeled as intervals of the real line. I provide a systematic examination of the methods by which these interval judgments can be combined.

The legal problem arises in the context of "community standards," legal standards of behavior rooted in the individual judgments or decisions of large numbers of people.[2] In this context, there are no preferences to aggregate; nor is there a conscious effort on the individuals in the community to delegate authority. Rather, courts judge the permissibility of actions according to whether the action is considered acceptable in the community.

Miller (2013) defines a model of community standards and provides conditions under which an aggregate of individual standards will deem an action impermissible when, and only when, all individuals in the community consider it to be impermissible. Because communities are generally understood to be large and diverse, this is understood to be a negative result. That model, however, made a strong assumption: that there was no natural structure on the set of alternatives upon which the aggregation method could be conditioned.

This paper weakens this assumption by providing a plausible structure on the set of alternatives. In particular, alternatives are assumed to be intervals of the real line. I show that this change in assumptions allows a broader set of aggregation methods, including one which corresponds to majority rule. This model thus integrates two important features from the prior literature—the choice of intervals and the study of

[2]Courts justify their decisions through reference to community standards in many areas of law, including negligence (Miller and Perry, 2012), contract (Miller and Perry, 2013a), defamation (Miller and Perry, 2013b), and obscenity (Miller, 2013).

aggregation—and allows the legislative and legal problems to be studied in a unified framework.

Having described the problem, I now proceed to describe the model. The first element of the model is the set of alternatives, which is isomorphic to the real line. Next, there are agents; these represent the members of the legislature, or of the community in the case of legal standards. Each agent has a set of allowable alternatives; this set is assumed assumed to be bounded (so that the set of allowable alternatives is limited) and convex (so that alternatives directly in between allowable alternatives are necessarily allowed).[3] In other words, judgments take the form of intervals of the real line. These judgments are then aggregated to form a collective judgment. In the legislative case, the collective judgment represents the discretion delegated by the legislature. In the legal case, the collective judgment represents the community standard.

The focus of this paper is on the method through which these judgments are combined. A set of axioms is defined and used (in various combinations) to characterize a new family of aggregation rules, which I term "endpoint rules." Endpoint rules aggregate the lower and upper endpoints separately, in a way that guarantees that that aggregate will be an interval.

These rules are parametrized by two positive integers, p and q, such that the sum of the two parameters is not greater than one more than the number of agents. As each agent has a judgment that takes the form of an interval, we can define the **p,q**-th endpoint rule as the one that defines the aggregate set to be the interval defined by the p-th lowest lower endpoint and the q-th highest upper endpoint. The subclass of rules where $p = q$ are called "symmetric" endpoint rules.

The family of endpoint rules includes the "median rule," in which $p = q = \lfloor \frac{n+1}{2} \rfloor$ (See Block, 2010; Miller, 2009). The lack of a median is a problem in both the legal and legislative contexts. In the legal setting, there may not be a well-defined median judgment. In a legislative body, there may neither be a median voter nor agreement as to the interval to be delegated. Endpoint rules provide one answer to this problem: as in Lax (2007), the median rule is well-defined even though a median voter or judgment may not exist.

I characterize the class of endpoint rules using several axioms. Two are direct analogues of axioms found in Miller (2013). The *responsiveness* axiom requires the aggregation rule to respond to changes in the individual judgments. If the individual interval changes, and each new interval includes the prior one, then responsiveness requires the new aggregate interval to include the older aggregate interval. The *anonymity* axiom requires the aggregate to be independent of the names of the agents, so that the aggregate choice would not change were two agents to trade their standards between themselves. The *continuity* axiom restricts small changes in the individual intervals from leading to a large change in the aggregate interval.

The fourth and fifth axioms are weaker versions of the neutrality axiom used

[3]To simplify results, I assume that judgments take the form of open sets.

in Miller (2013). To understand these axioms, it is important to recall that this paper imposes structure on the set of alternatives, in that the set of alternatives is isomorphic to the real line. In particular, there are two properties of the real line that are important in this paper: betweenness and direction. I illustrate these through a simple example.

Suppose we were to ask: at what speeds would it be reasonable to drive on the autobahn on a clear day with light traffic? Different individuals would have different answers to this question. However, the judgments should all have one property in common. If Alice believes that it is permissible to drive at 100 km/hour, and that it would also believe it to be permissible to drive at 140 km/hour, then she should consider it reasonable to drive at 120 km/hour as well. There is a natural structure on the set of possible speeds: some speeds are objectively in-between other speeds. (See Nehring and Puppe, 2002, 2007, for more on betweenness.) As each individual's set of reasonable speeds is convex, the aggregate set of reasonable speeds should also be convex. Furthermore, the natural structure has another important property: 140 km/hour is objectively faster than 100 km/hour. In other words, the direction (high relative to low) matters.

The concern with betweenness and direction leads to two versions of neutrality. First, *weak neutrality*, requires the aggregation of individual judgments to be independent of transformations of the real line that preserve both betweenness and the direction. It does not require the aggregation to be independent of transformations that preserve betweenness only. This property implies that the cardinality properties of the real line should be disregarded. Weak neutrality is equivalent to the ordinal covariance axiom of Chambers (2007).

In some contexts, betweenness is relevant but direction is not. While a "centrist" political position is objectively in between a "left wing" and a "right wing" position, the identification of one extreme with the left and the other with the right is the result of an arbitrary convention. *Strong neutrality*, as its name implies, is stronger than weak neutrality. It states that the aggregation of individual judgments must be independent of any betweenness-preserving transformation of the real line. This property implies that betweenness is important, but that the direction and cardinality properties of the real line should be disregarded.

Using these axioms, I prove two results. First, every endpoint rule satisfies the responsiveness, anonymity, continuity, and weak neutrality axioms; furthermore, any aggregation rule that satisfies these axioms is an endpoint rule. Second, replacing weak neutrality with strong neutrality yields a characterization of the symmetric endpoint rules.

1.1 Strategyproofness

The interval aggregation problem described above does not use the concept of preference. In the legal context, individuals' judgments represent the individuals' beliefs about which actions are acceptable, and not their preferences over policy (for more on this distinction, see Kornhauser and Sager, 1986). In the legislative context, the intervals are derived from the legislators' preferences, but do not represent them directly. The model takes the judgments as given, and does not ask where they come from.

However, even though we may be opposed to strategic judgments in some contexts as a matter of principle, this does not mean that strategic judgments are never made. For this reason, one may wish to know the extent to which the endpoint rules manipulable? To answer this question, I investigate the implications of a *strategyproofness* assumption (see Dummett and Farquharson, 1961).

To study the implications of strategyproofness it is necessary to restrict the class of allowable preferences (see Gibbard, 1973; Satterthwaite, 1975), and it is known that, when choosing a single alternative from a single issue dimension, a voting rule can be strategyproof and non-dictatorial if preferences are single-peaked (see Moulin, 1980). Block (2010) defines a betweenness relation on intervals and uses this definition to formulate a class of generalized single-peaked preferences (see Nehring and Puppe, 2008). According to this definition, an interval is defined to be between two other intervals if its lower endpoint is between the lower endpoints of the other two, and if its upper endpoint is between the upper endpoints of the other two. Preferences are generalized single-peaked if there is (a) a unique interval that is preferred to all other intervals (called the "peak") and (b) any interval in between the peak and a third interval is necessarily preferred to the third interval. An aggregation rule is strategyproof if each individual prefers to truthfully reveal his or her peak interval. Block (2010) showed that the median rule is strategyproof.

I show that all endpoint rules are strategyproof. Furthermore, I show that all strategyproof aggregation rules are responsive and continuous. This implies that any aggregation rule satisfying anonymity, strategyproofness, and neutrality must be an endpoint rule.

1.2 Independent Aggregation of Endpoints

Endpoint rules do not aggregate judgments in a pointwise manner. Instead, these rules aggregate judgments according to their endpoints. Furthermore, they aggregate the endpoints independently. That is, the aggregate left endpoint is a determined without reference to the individual right endpoints, and vice versa.

To illustrate, consider the question from before: at what speeds would it be reasonable to drive on the autobahn on a clear day with light traffic? A traditional way to answer this question would be to aggregate each speed independently, in a pointwise manner. That is, for each speed, we would ask whether a majority would consider it reasonable to drive at that speed. However, an implication of this analysis is that such pointwise aggregation may lead to incoherent outcomes as it will not necessarily lead to a well-defined interval. There may be no speed that a majority considers to be reasonable. If we lower the threshold (below a majority), we may find that the set of reasonable speeds is not convex.

In practice, of course, this incoherence will not be apparent when it arises, as a court will not attempt to aggregate decisions about all such speeds. Instead, such a court will only consider the case in front of it. Regardless of whether the court decides driving at 120 km/hour to be reasonable, or not, an observer will not receive enough information to tell whether similar cases would have been decided in a coherent manner. But the problem of incoherence still remains, and the decision would still be fundamentally arbitrary.

Endpoint rules solve this problem by separating the decision of whether an action is permissible into two questions: First, is there a 'lesser' permissible action? Second, is there a 'greater' permissible action? That is, we first check to see whether a majority believes it permissible to drive at a speed below 120 km/hour, and we then check to see whether a majority considers it permissible to drive at a speed above 120 km/hour. The rule decides that this speed is permissible if, and only if, the answers to both of these questions is yes.

I show that the property of independent aggregation of endpoints follows directly from the strategyproofness axiom. As a consequence, the aggregation of lower endpoints is essentially equivalent to the aggregation of single peaked preferences on a single issue dimension as in Moulin (1980). I provide an analogue of Moulin's "phantom voters" characterization to characterize an analogous family of rules that includes the endpoint rules as a special case.

1.3 Other literature

There is a significant literature devoted to the study of opinion and judgment aggregation, starting with the pioneering works of Arrow (1963) and May (1952). A number of papers study the aggregation of sets, with differing interpretations. Ahn and Chambers (2010), for example, study menu choice, while Miller (2013) uses a similar model to study an important class of legal standards. Similar results, using different sets of axioms, can also be derived from Monjardet (1990) and Nehring and Puppe (2007).

There is a conceptual link between the results in this paper and those of Chambers (2007), which characterizes quantile representations using ordinal covariance and monotonicity. Ordinal covariance is essentially weak neutrality, while monotonicity is closely related to responsiveness. Endpoint rules can be thought of as a type of a quantile rule, where each endpoint is chosen according to a quantile. A contribution of the present work is that endpoint rules allow for independent and consistent aggregation of the two endpoints.

2 Endpoint Rules

Let $N \equiv \{1, \ldots, n\}$ be a finite set of agents, and let Σ be the set of bounded open convex intervals of the real line. I wish to study aggregation functions $f : \Sigma^N \to \Sigma$, which map a set of n intervals into a single interval. The responsiveness and anonymity axioms used in Miller (2013) can be defined in this environment.

Responsiveness: For all $S, T \in \Sigma^N$, if $S_i \subseteq T_i$ for all $i \in N$, then $f(S) \subseteq f(T)$.

Let π denote a permutation of N, and define $\pi S = \big(S_{\pi(1)}, \ldots, S_{\pi(n)}\big)$.

Anonymity: For every π of N and $S \in \Sigma^N$, $f(S) = f(\pi S)$.

I also introduce a basic continuity axiom. For two intervals $S_i, T_i \in \Sigma$, define $d(S_i, T_i) \equiv \mu((S_i \cup T_i) \setminus (S_i \cap T_i))$.

Continuity: For every $\delta > 0$ there exists an $\varepsilon > 0$ such that for all $S, T \in \Sigma^N$, $d(S_i, T_i) \leq \varepsilon$ for all $i \in N$ implies that $d(f(S), f(T)) \leq \delta$.

The neutrality axiom used in Miller (2013) must be modified to take into account of the natural structure. I suggest two distinct neutrality axioms. Let Φ be the set of all strictly monotone transformations of the real line, and let Φ^+ be the set of all strictly increasing monotone transformations of the real line. That is, transformations in Φ must preserve betweenness, while transformations in Φ^+ must additionally preserve the direction. In neither set, however, are transformations required to preserve the cardinal properties of the real line. For $S_i \in \Sigma$ and $\phi \in \Phi$, define $\phi(S_i) \equiv \cup_{x \in S_i} \phi(x)$.

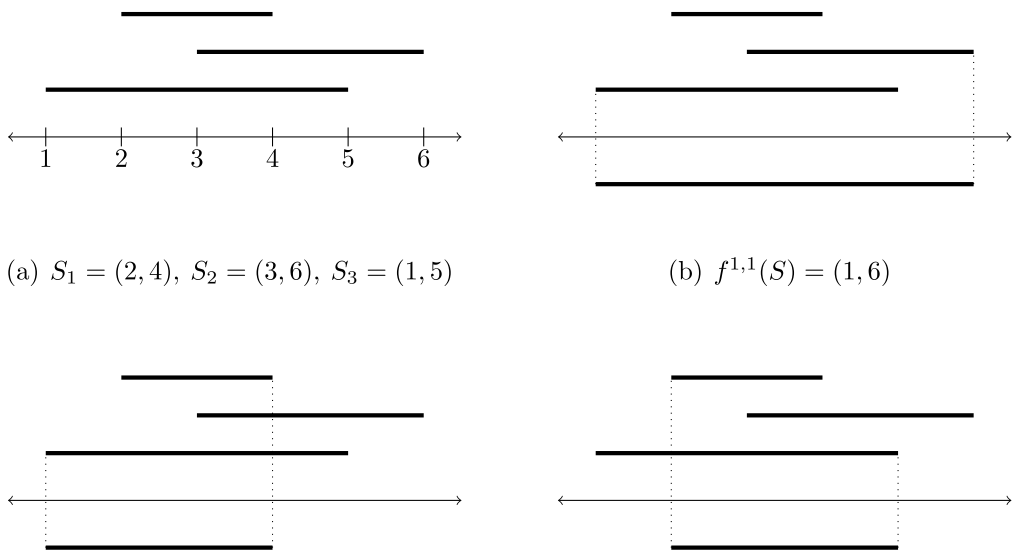
Weak Neutrality: For every $\phi \in \Phi^+$ and $S \in \Sigma^N$, $\phi(f(S)) = f(\phi(S_1), \ldots, \phi(S_n))$.

Strong Neutrality: For every $\phi \in \Phi$ and $S \in \Sigma^N$, $\phi(f(S)) = f(\phi(S_1), \ldots, \phi(S_n))$.

I introduce an interesting class of aggregation rules, called endpoint rules, which I believe have not yet been described in the literature. This class of rules is parameterized by two positive integers, p and q, such that $p + q \leq n + 1$. For a profile of standards S and a point $x \in \mathbb{R}$, let $G^+(S, x) = \{i \in N : (-\infty, x] \cap S_i \neq \varnothing\}$ be the set of individuals who have x, or a point below x, in their interval, and let $G^-(S, x) = \{i \in N : [x, +\infty) \cap S_i \neq \varnothing\}$ who have x, or a point above x, in their interval. An endpoint rule is of the form

$$f^{p,q}(S) \equiv \{x : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}.$$

This rule takes the open interval defined by the p-th lowest lower endpoint and the q-th highest upper endpoint. For example, three individuals are depicted in Figure 1(a), with standards $S_1 = (2, 4)$, $S_2 = (3, 6)$, and $S_3 = (1, 5)$. Here, $f^{1,1}(S) = (1, 6)$ (Figure 1(b)), $f^{1,3}(S) = (1, 4)$ (Figure 1(c)), and $f^{2,2}(S) = (2, 5)$ (Figure 1(d)).



(a) $S_1 = (2,4),\ S_2 = (3,6),\ S_3 = (1,5)$

(b) $f^{1,1}(S) = (1,6)$

(c) $f^{1,3}(S) = (1,4)$

(d) $f^{2,2}(S) = (2,5)$

Figure 1: Endpoint Rules

An important subclass of rules is that of the symmetric endpoint rules, where $p = q$. If we define m to be the maximal integer less or equal to $(n+1)/2$, then $f^{m,m}(S)$ is the median rule (See Block, 2010; Miller, 2009).

I present two results. The first theorem is a characterization of the endpoint rules.

Theorem 1. *An aggregation rule f satisfies responsiveness, anonymity, continuity, and weak neutrality if and only if it is an endpoint rule.*

The second theorem is a characterization of symmetric endpoint rules.

Theorem 2. *An aggregation rule f satisfies responsiveness, anonymity, continuity, and strong neutrality if and only if it is a symmetric endpoint rule.*

The proofs are in the appendix. The sets of axioms used in both theorems are independent when $n \geq 3$; however, symmetric endpoint rules can be characterized without continuity in the case where $n = 2$.[4]

3 Strategyproofness

Block (2010) shows that the median rule is strategyproof. In this section I extend this result to show that all endpoint rules are strategyproof. To study the question

[4]The proofs of these facts are left as an exercise for the reader.

of strategyproofness it is necessary to make an assumption about preferences.[5] Block (2010) defines a class of single-peaked preferences on intervals that relies on a concept of betweenness. An interval is defined to be between two other intervals if its lower endpoint is between the lower endpoints of the other two, and if its upper endpoint is between the upper endpoints of the other two. For intervals $R, T \in \Sigma$, let $\mathcal{B}(R,T) \subset \Sigma$ be the set such that $S \in \mathcal{B}(R,T)$ if

$$\inf(R) \leq \inf(S) \leq \inf(T) \textbf{ or } \inf(R) \geq \inf(S) \geq \inf(T) \tag{1}$$

and

$$\sup(R) \leq \sup(S) \leq \sup(T) \textbf{ or } \sup(R) \geq \sup(S) \geq \sup(T). \tag{2}$$

Single-peaked preferences are preferences for which (1) there is a unique preferred interval (the 'peak') and (2) an interval that is between the peak and a third interval is preferred to that third interval. Let $\mathcal{P}$ be the set of preferences on Σ such that, for all $\succeq_i \in \mathcal{P}$, (1) there exists $S_i^* \in \Sigma$ such that $T \succeq_i S_i^*$ implies that $T = S_i^*$ and (2) for $R, T \in \Sigma$, $R \in \mathcal{B}(S_i^*, T)$ implies that $S_i^* \succeq_i R \succeq_i T$.

An aggregation rule is strategyproof if it is always in an agent's interest to reveal her preferred interval, holding the other judgments constant.

Strategyproofness: For every $S \in \Sigma^N$, $i \in N$, and $\succeq_i \in \mathcal{P}$, $f(S_1, ..., S_i^*, ..., S_n) \succeq_i f(S)$.

Strategyproofness implies the responsiveness and continuity axioms.

Proposition 1. *An aggregation rule f satisfies strategyproofness only if it is responsive and continuous.*

Furthermore, all endpoint rules are strategyproof.

Proposition 2. *Endpoint rules are strategyproof.*

The proofs of these propositions are in the appendix.[6] These propositions enable characterizations of the endpoint rules in terms of anonymity, neutrality, and strategyproofness. The sets of axioms used in the corollaries are independent.

Corollary 1. *An aggregation rule f satisfies anonymity, strategyproofness, and weak neutrality if and only if it is an endpoint rule.*

Corollary 2. *An aggregation rule f satisfies anonymity, strategyproofness, and strong neutrality if and only if it is a symmetric endpoint rule.*

[5]No assumption about preferences has been made up to this point, as judgments need not come from a preference ordering

[6]In a finite setting, strategyproofness of endpoint rules can be proven using Nehring and Puppe (2008, Theorem 3).

3.1 Independent Aggregation of Endpoints

An important property of endpoint rules is that the endpoints are aggregated independently. The property can be defined formally as follows:

Independent aggregation of endpoints: For every $S, T \in \Sigma^N$ such that $\inf(S_i) = \inf(T_i)$ for all $i \in N$, $\inf(f(S)) = \inf(f(T))$. For every $S, T \in \Sigma^N$ such that $\sup(S_i) = \sup(T_i)$ for all $i \in N$, $\sup(f(S)) = \sup(f(T))$.

It turns out that all strategyproof rules have this property; that is, if a rule satisfies strategyproofness then it must aggregate the lower endpoints without consider the upper endpoints, and vice versa. This is stated in the following proposition.

Proposition 3. *An aggregation rule f satisfies strategyproofness only if it satisfies independent aggregation of endpoints.*

The proof of this proposition is in the appendix.

3.1.1 Connection to Moulin (1980)

A consequence of Proposition 3 is that, under the assumption of strategyproofness, the aggregation of lower (and upper) endpoints is essentially equivalent to the aggregation of single-peaked preferences on a single issue dimension, as studied in Moulin (1980). Using this insight, I provide an analogue to Moulin's characterization of "phantom voters" that includes the endpoint rules as a special case.

Define $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, \infty\}$ as the extended reals, and define $\overline{<}$ as a binary relation on $\overline{\mathbb{R}}$ such that, for $x, y \in \overline{\mathbb{R}}$, $x \overline{<} y$ if (i) $x, y \in \mathbb{R}$ and $x < y$, (ii) $x = -\infty$, or (iii) $y = \infty$. Let $\overline{\Sigma} = \{(x, y) \in \overline{\mathbb{R}} : x \overline{<} y\}$ be the open convex intervals of the extended reals. Note that $\Sigma \subset \overline{\Sigma}$. Let $\textsc{med} : \overline{\Sigma}^{2n+1} \to \overline{\Sigma}$ be the median rule applied to $2n+1$ intervals of the extended reals; that is, for a profile $Q \in \overline{\Sigma}^{2n+1}$, $\textsc{med}(Q) = \{x \in \mathbb{R} : \min\{|\{i : (-\infty, x] \cap Q_i \neq \varnothing\}|, |\{i : [x, +\infty) \cap Q_i \neq \varnothing\}|\} \geq n+1\}$.

I establish the following claim. The proof relies on Moulin (1980) and is left as an exercise for the reader.

Claim 1. *An aggregation rule f satisfies anonymity and strategyproofness if there exists $P \in \overline{\Sigma}^{n+1}$ such that for all $S \in \Sigma^N$, $f(S) = \textsc{med}(S, P)$.*

Here, the elements $P \in \overline{\Sigma}$ may be thought of as "phantom intervals;" unlike the regular intervals, however, these include the half-bounded intervals of the form $(-\infty, x)$ and (x, ∞) for $x \in \mathbb{R}$, and the fully unbounded intervals of the form $(-\infty, -\infty)$, $(-\infty, \infty)$, and (∞, ∞).

If imposed, weak neutrality would eliminate half-bounded intervals; thus all phantom intervals would be fully unbounded. By Corollary 1, weak neutrality would also imply that the resulting rules are endpoint rules. The connection can be easily observed: the p, q-th endpoint rule is characterized by p phantoms of the form (∞, ∞),

q phantoms of the form $(-\infty, -\infty)$, and $n+1-p-q$ phantoms of the form $(-\infty, \infty)$. Strong neutrality would additionally imply that the number of intervals of the form $(-\infty, -\infty)$ must equal the number of the form (∞, ∞), and therefore that $p = q$.

4 Conclusion

I have introduced the endpoint rules and have shown that they are characterized by responsiveness, anonymity, continuity, and neutrality in this setting. Furthermore, I have shown that with a suitable restriction on preferences, endpoint rules are strategyproof, and that all strategyproof, anonymous, and neutral aggregation rules are endpoint rules.

One may ask whether more general results can be established by focusing on the abstract properties of the betweenness relation and the order. To provide a short example: consider the case of a decision that must be made on two (or more) dimensions. We must decide not only how fast it is reasonable to drive, but also, how much training drivers should have before getting behind the wheel. There may be a tradeoff; at higher speeds, more training is necessary, although different individuals may have different views about the right tradeoff. This is a much more complex question, and it is difficult to study. In the case of the real line, the concept of betweenness implies intervals, which can be identified with points in two-dimensional space (such that $x_1 < x_2$). In the case of multidimensional space, however, betweenness simply implies convexity, and there is no similarly easy way to represent these convex sets. In addition, there may be interesting problems with different underlying structures, for which the simple assumption of Euclidean space may not be applicable.

Future research may investigate the similarities between the legal and legislative contexts considered in this paper. Loosely speaking, one may think of a law that relies on a community standards (such as that permitting reasonable behavior) as one that delegates a limited form of discretion to the actor. The actor is not acting on behalf of a principal, but the motivation is nonetheless similar. the actor is better informed (at least as circumstances surrounding the action), but discretion is constrained so as to limit the actor's ability to harm society (through "unreasonable" actions).[7]

There are important differences, of course, between these models. In case of optimal discretion, the bounds are explicitly set by the principal and are known to the agent *ex ante*, while in the case of reasonableness, the bounds are implicitly set by a long series of judicial decisions that require the actor to behave according to a community standard. Furthermore, the determination of whether the actor has complied with the community standard is made by a court *ex post*. "Endpoint rules" provide a means by which this determination can be made, and a justification of judicial decisions as something more than an arbitrary decision of a court.

[7] It is by no means agreed upon that the law tries to maximize an ordering. The model in this paper is general enough to cover non-welfarist theories of law.

Appendix

I first state and prove the following lemma.

Lemma 1. *If f satisfies anonymity and weak neutrality, then for every $S, T \in \Sigma^N$, every permutation π of N, and every $\phi \in \Phi^+$ such that $\pi S = \phi T$, if there is an endpoint rule $f^{p,q}$ such that $f(S) = f^{p,q}(S)$, then $f(T) = f^{p,q}(T)$.*

Proof of Lemma 1. Let $S, T \in \Sigma^N$, and let π be a permutation of N and $\phi \in \Phi^+$ such that $\pi S = \phi T$. Let f satisfy anonymity and weak neutrality, and let $f^{p,q}$ be an endpoint rule such that $f(S) = f^{p,q}(S)$. Note that by the definition of the endpoint rule, $f^{p,q}(S) = \phi f^{p,q}(T)$. By anonymity, $f(S) = f(\pi S) = f(\phi T)$. By weak neutrality, $f(\phi T) = \phi f(T)$, and therefore $f(S) = \phi f(T)$. Because ϕ is strictly monotone there exists an inverse $\phi^{-1} \in \Phi$ such that $\phi^{-1}\phi S = S$; therefore $\phi^{-1} f(S) = f(T)$ and $\phi^{-1} f^{p,q}(S) = f^{p,q}(T)$. Because $f(S) = f^{p,q}(S)$ it follows that $f(T) = f^{p,q}(T)$. $\square$

Proof of Theorem 1. That endpoint rules satisfy the four axioms is trivial. Let f satisfy the four axioms. I show that f must be an endpoint rule. For $S \in \Sigma^N$ and $p, q \leq n$, define the function $f^{p,q}(S) \equiv \{x : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}$, and define $Q(S) \equiv \{(p, q) \in N^2 : f(S) = f^{p,q}(S)\}$. I will show that there exists $p, q \leq n$, where $p + q \leq n + 1$, such that $(p, q) \in Q(S)$ for all $S \in \Sigma^N$.

Part One: I show that $|Q(S)| \geq 1$ for all $S \in \Sigma^N$.

For $S \in \Sigma^N$ define $L(S) \equiv \cup_i \inf(S_i)$ and $U(S) \equiv \cup_i \sup(S_i)$. It is sufficient to show that $\inf(f(S)) \in L(S)$ and $\sup(f(S)) \in U(S)$.

First, I show that for all $S \in \Sigma^N$, $\inf(f(S)), \sup(f(S)) \in L(S) \cup U(S)$. To see this, let $S \in \Sigma^N$ and suppose, contrariwise, that $\inf(f(S)) \notin L(S) \cup U(S)$. Let $\phi \in \Phi^+$ such that $\phi(\inf(f(S))) \neq \inf(f(S))$ and, for all $i \in N$, $\phi(\inf(S_i)) = \inf(S_i)$ and $\phi(\sup(S_i)) = \sup(S_i)$. Then $S = \phi S$, so $f(S) = f(\phi S)$. By weak neutrality, $f(\phi S) = \phi(f(S))$ and therefore, $f(S) = \phi f(S)$. It follows that $\inf(f(S)) = \inf(\phi f(S)) = \phi(\inf(f(S)))$, a contradiction.

Next, I show that for all $S \in \Sigma^N$, $\inf(f(S)) \in L(S)$ and $\sup(f(S)) \in U(S)$. Suppose, contrariwise, that this is false, and assume, without loss of generality, that $\inf(f(S)) \notin L(S)$. Because $\inf(f(S)) \notin L(S)$, it must be that $\inf(f(S)) \in U(S)$. Therefore, there exists a group $M \subseteq N$, $M \neq \varnothing$, such that $\inf(f(S)) = \sup(S_j)$ for all $j \in M$.

Let $\varepsilon > 0$ such that, for all $i \in N$, $\inf(f(S)) \geq \inf(S_i)$ if and only if $\inf(f(S)) + \varepsilon \geq \inf(S_i)$, and, for all $j \in N \setminus M$, $\inf(f(S)) \geq \sup(S_i)$ if and only if $\inf(f(S)) + \varepsilon \geq \sup(S_i)$.

Let $\phi \in \Phi^+$ such that (i) for all $i \in N$, $\phi(\inf(S_i)) = \inf(S_i)$, (ii) for all $j \in N \setminus M$, $\phi(\sup(S_j)) = \sup(S_j)$, and (iii) $\phi(\inf(f(S))) = \inf(f(S)) + \varepsilon$.

Let $S' \in \Sigma^N$ such that, for all $j \in N \setminus M$, $S'_j = S_j$ and, for all $k \in M$, $S'_k = (\inf(S_k), \sup(S_k) + \varepsilon)$. Because $S_i \subseteq S'_i$ for all $i \in N$ it follows that $f(S) \subseteq f(S')$, and therefore that $\inf(f(S)) \geq \inf(f(S'))$. Because $\phi S = S'$ it follows that $f(\phi S) = f(S')$, and by responsiveness that $f(\phi S) = \phi f(S) = \phi f(S)$. Hence $\phi f(S) = f(S')$, and therefore $\inf(f(S)) + \varepsilon = \inf(f(S'))$. This implies that $\inf f(S) < \inf f(S')$, a contradiction.

Part Two. For $k \in N$, let $S^k \in \Sigma^N$ such that, for all $i < k$, $S_i^k = (2i-1, 2i)$, and for all $j \geq k$, $S_j^k = (j+k-1, j+n)$. I prove that $Q(S^1) = Q(S^n)$. It is sufficient to show that $Q(S^k) = Q(S^{k+1})$ for all $k \in N \setminus \{n\}$.

Let $k \in N \setminus \{n\}$. From Part One we know that $|Q(S^k)| \geq 1$. Because $f^{p,q}(S^k) = f^{p',q'}(S^k)$ only if $p' = p$ and $q' = q$, it follows that $|Q(S^k)| \leq 1$. Thus, $|Q(S^k)| = 1$.

By continuity there exists $\varepsilon > 0$ such that $d(S_i, T_i) < 3\varepsilon$ for all $i \in N$ implies that $d(f(S), f(T)) < 0.1$.

For $\ell \in \{1, \ldots, n-k\}$, let $T^{\ell,+}, T^{\ell,-} \in \Sigma^N$ such that $T_k^{\ell,+} = (2k-1, n+k-\ell+\varepsilon)$, $T_k^{\ell,-} = (2k-1, n+k-\ell-\varepsilon)$, and for $i \neq k$, $T_i^{\ell,+} = T_i^{\ell,-} = S_i^k$.

First, I show that for $\ell \in \{1, \ldots, n-k\}$, $Q(T^{\ell,+}) = Q(T^{\ell,-})$. To see this, note that $d(T_k^{\ell,+}, T_k^{\ell,-}) = 2\varepsilon$, and for $i \neq k$, $d(T_i^{\ell,+}, T_i^{\ell,-}) = 0$. It follows that $d(f(T^{\ell,+}), f(T^{\ell,-})) < 0.1$. Note that $d(f^{p,q}(T^{\ell,+}), f^{p',q'}(T^{\ell,-})) > 0.1$ unless $p' = p$ and $q' = q$. This implies that (a) $Q(T^{\ell,+}) = Q(T^{\ell,-})$.

Next, let $\phi^0 \in \Phi^+$ such that (i) $\phi^0(n+k) = n+k-1+\varepsilon$ and (ii) for all $x \in \mathbb{N} \setminus \{n+k\}$, $\phi^0(x) = x$. Because $\phi^0 S^k = T^{1,+}$, it follows as a consequence of Lemma 1 that (b) $Q(S^k) = Q(T^{1,+})$.

For $\ell \in \{1, \ldots, n-k-1\}$, let $\phi^\ell \in \Phi^+$ such that (i) $\phi^\ell(n+k-\ell-\varepsilon) = n+k-\ell-1+\varepsilon$ and (ii) for all $x \in \mathbb{N}$, $\phi^\ell(x) = x$. Because $\phi^\ell T^{\ell,-} = T^{\ell+1,+}$, it follows as a consequence of Lemma 1 that (c) $Q(T^{\ell,-}) = Q(T^{\ell+1,+})$.

Let $\phi^{n-k} \in \Phi^+$ such that (i) $\phi^{n-k}(2k-\varepsilon) = 2k$, (ii) for all $i \in \{k+1, \ldots, n\}$, $\phi^{n-k}(i+k-1) = i+k$, and (iii) for all $x \in \mathbb{N} \setminus \{2k, \ldots, n+k\}$, $\phi^{n-k}(x) = x$. Because $\phi^{n-k} T^{n-k,-} = S^{k+1}$, it follows as a consequence of Lemma 1 that (d) $Q(T^{n-k,-}) = Q(S^{k+1})$.

By combining (a), (b), (c), and (d), we have that $Q(S^k) = Q(S^{k+1})$.

Part Three. Let $\dot{p}, \dot{q} \leq n$ such that $f(S^1) = f^{\dot{p},\dot{q}}(S^1)$. I prove that for all $S \in \Sigma^N$, $f(S) = f^{\dot{p},\dot{q}}(S)$.

First, I show that $f(S) \subseteq f^{\dot{p},\dot{q}}(S)$. Suppose that this is false. Then by part one, $f(S) = f^{p',q'}(S)$, where either $p' < \dot{p}$ or $q' < \dot{q}$. Without loss of generality, assume that $p' < \dot{p}$. Let $x \in f(S)$ such that $x < \inf f^{\dot{p},\dot{q}}(S)$.

Let π be a permutation of N such that, for all $i, j \in N$, $\inf(S_i) < \inf(S_j)$ implies that $\pi(i) < \pi(j)$. Observe that $\inf(f^{1,1}(S)) \leq \inf S_{\pi^{-1}(p')} < x < \inf S_{\pi^{-1}(p*)} < \sup(f^{1,1}(S))$. Let $\phi \in \Phi^+$ such that (a) $\phi \inf(f^{1,1}(S)) > \dot{p}-1$, (b) $\phi \inf S_{\pi^{-1}(p')} = \dot{p} - \frac{1}{2}$, (c) $\phi x = \dot{p} - \frac{1}{4}$, (d) $\phi \inf S_{\pi^{-1}(p'*} > n$, and (e) $\phi \sup(f^{1,1}(S)) < n+1$.

Note that $\phi S_i \subseteq \pi S_i^1$ for all $i \in N$. To see that this is true, observe that for j such that $\pi(j) < \dot{p}$, $\inf(\phi S_j) > \dot{p} - 1 \geq \pi(j) = \inf(S_{\pi(j)}^1)$ and $\sup(\phi S_j) < n+1 \leq n + \pi(j) = \sup(S_{\pi(j)}^1)$, and for j such that $\pi(j) \geq \dot{p}$, $\inf(\phi S_j) > n \geq \pi(j) = \inf(S_{\pi(j)}^1)$ and $\sup(\phi S_j) < n+1 \leq n+\pi(j) = \sup(S_{\pi(j)}^1)$.

Because f satisfies responsiveness and anonymity, $\phi f(S) \subseteq f(S^1)$. Because $\inf f^{\dot{p},\dot{q}}(S) \in f(S)$ it follows that $\phi \inf f^{\dot{p},\dot{q}}(S) \in \phi f(S)$. By construction, $\phi \inf f^{\dot{p},\dot{q}}(S) = \dot{p}$. But this implies that $\dot{p} \in f(S^1)$, a contradiction.

Next, I show that $f^{\dot{p},\dot{q}}(S) \subseteq f(S)$. Let $x \in f^{\dot{p},\dot{q}}(S)$. I show that $x \in f(S)$.

For $i \in N$, choose $x_i \in \mathbb{R}$ such that (a) $x_i \in S_i$, (b) $x_i \neq x_j$ for $j \neq i$, (c) $|\{i \in N : x_i \leq x\}| = \dot{p}$, and (d) $|\{i \in N : x_i \geq x\}| = \dot{q}$. Let $\varepsilon > 0$ such that (i) for all $i \in N$, $(x_i - \varepsilon, x_i + \varepsilon) \in S_i$), and (ii) $\varepsilon < \min_{i,j} |x_i - x_j|$. Define $X \in \Sigma^N$ such that $X \equiv (x_i - \varepsilon, x_i + \varepsilon)$.

Let π' be a permutation of N such that, for all $i, j \in N$, $x_i < x_j$ implies that $\pi(i) < \pi(j)$. Let $\phi \in \Phi^+$ such that for all $i \in N$, $\phi(x_i - \varepsilon) = 2\pi(i) - 1$ and $\phi(x_i + \varepsilon) = 2\pi(i)$. Note that $\phi(x) > 2\dot{p} - 1$ and that $\phi(x) < 2(n + 1 - \dot{q})$.

Note that $\pi\phi X = S^n$, which implies that $f(\pi\phi X) = f(S^n) = (2\dot{p}-1, 2(n+1-\dot{q}))$. By neutrality and anonymity, it follows that $\phi f(X) = (2\dot{p} - 1, 2(n + 1 - \dot{q}))$ which implies that $f(X) = (x_{\dot{p}} - \varepsilon, x_{n+1-\dot{q}} + \varepsilon)$, and hence, $x \in f(X)$. Because $X_i \subseteq S_i$ for all $i \in N$, it follows that $x \in f(S)$.

Part Four: The last step is to show that $\dot{p} + \dot{q} \leq n + 1$. Suppose contrariwise that $\dot{p} + \dot{q} > n + 1$. Then $\dot{p} - 1 \geq n + 1 - \dot{q}$. Thus $\inf(f(S^n)) = \inf(S^n_{\dot{p}}) = 2\dot{p} - 1$ and $\sup(f(S^n))) = \sup(S^n_{n+1-\dot{q}}) = 2(n + 1 - \dot{q})$. Because $\dot{p} - 1 \geq n + 1 - \dot{q}$ it follows that $2\dot{p} - 1 > 2(\dot{p} - 1) \geq 2(n + 1 - \dot{q})$. This implies that $\inf(f(S^n))) > \sup(f(S^n)))$, a contradiction. □

Proof of Theorem 2. That symmetric endpoint rules satisfy the axioms is trivial. Let f satisfy the axioms. Because strong neutrality implies weak neutrality, f is an endpoint rule with quotas p and q. I show that $p = q$. It is sufficient to show that for all $S \in \Sigma^N$, $f^{p,q}(S) = f^{q,p}(S)$.

Let $S \in \Sigma^N$ and let $\phi \in \Phi$ be the transformation such that $\phi(x) = -x$ for all $x \in \mathbb{R}$. Note that $f^{p,q}(S) = f^{p,q}(\phi\phi S)$ and, by strong neutrality, that $f^{p,q}(\phi\phi S) = \phi f^{p,q}(\phi S)$. It remains to be shown that $f^{p,q}(\phi(S)) = \phi(f^{q,p}(S))$. To see this, note that $\phi(f^{p,q}(S)) = \{\phi(x) : |G^+(S, x)| \geq p \text{ and } |G^-(S, x)| \geq q\}$. Because $\phi = \phi^{-1}$, it follows that $\phi(f^{p,q}(S)) = \{x : |G^+(S, \phi(x))| \geq p \text{ and } |G^-(S, \phi(x))| \geq q\}$. Because ϕ is decreasing, $G^+(S, \phi(x)) = G^-(\phi(S), x)$ and $G^-(S, \phi(x)) = G^+(\phi(S), x)$. Hence, $\phi(f^{p,q}(S)) = \{x : |G^-(\phi(S), x)| \geq p \text{ and } |G^+(\phi(S), x)| \geq q\} = f^{q,p}(\phi(S))$. □

For two points $x, y \in \mathbb{R}$, let $B(x, y) = \{z \in \mathbb{R} : x \leq z \leq y \text{ or } y \leq z \leq x\}$.

Lower property: For $i \in N$ and $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$, either (a) $\inf(f(S)) = \inf(f(T))$ or (b) $\inf(f(S)) \in B(\inf(S_i), \inf(f(T)))$ and $\inf(f(T)) \in B(\inf(T_i), \inf(f(S)))$.

Upper property: For $i \in N$ and $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$, either (a) $\sup(f(S)) = \sup(f(T))$ or (b) $\sup(f(S)) \in B(\sup(S_i), \sup(f(T)))$ and $\sup(f(T)) \in B(\sup(T_i), \sup(f(S)))$.

I next state and prove the following lemma.

Lemma 2. *Strategyproofness implies both the lower property and the upper property.*

Proof. Let $i \in N$ and let $S, T \in \Sigma^N$ such that $S_j = T_j$ for all $j \neq i$. Let f satisfy strategyproofness.

Strategyproofness implies that for every preference $\succeq^S \in \mathcal{P}$ with peak S_i, $f(S) \succeq^S f(T)$. This implies that (i) $f(S) \in \mathcal{B}(S_i, f(T))$. A similar argument shows that (ii) $f(T) \in \mathcal{B}(T_i, f(S))$. To prove the lower property, there are three cases:

Case 1: $\inf(f(S)) \in B(\inf(S_i), \inf(T_i))$. Statement (ii) implies that $\inf(f(T)) \in B(\inf(T_i), \inf(f(S)))$, which implies that $\inf(f(S)) \in B(\inf(S_i), \inf(f(T)))$.

Case 2. $\inf(f(S)) > \inf(S_i), \inf(T_i)$. By statement (i) $\inf(f(S)) \leq \inf(f(T))$. By statement (ii) $\inf(f(T)) \leq \inf(f(S))$. This implies that $\inf(f(S)) = \inf(f(T))$.

Case 3. $\inf(f(S)) < \inf(S_i), \inf(T_i)$. By statement (i) $\inf(f(T)) \leq \inf(f(S))$. By statement (ii) $\inf(f(S)) \leq \inf(f(T))$. This implies that $\inf(f(S)) = \inf(f(T))$.

The upper property is proven in a similar fashion. □

Proof of Proposition 1. Let f satisfies strategyproofness. It follows from Lemma 2 that f satisfies the lower and upper properties.

Part One. I show that strategyproofness implies continuity.

Let $\delta > 0$ and let $\varepsilon = \frac{\delta}{n}$. Let $S, T \in \Sigma^N$ such that $d(S_i, T_i) \leq \varepsilon$ for all $i \in N$. For $j \in N$ let $S^j \in \Sigma^N$ such that $S_i^j = S_i$ for $i \leq j$ and such that $S_i^j = T_i$ otherwise. Define $S^0 \equiv S$. Let $k \in N$. I will show that $d(f(S^{k-1}), f(S^k)) \leq \varepsilon$.

Because $d(S_k^{k-1}, S_k^k) \leq \varepsilon$ it follows that $|\inf(S_k^{k-1}) - \inf(S_k^k)| + |\sup(S_k^{k-1}) - \sup(S_k^k)| \leq \varepsilon$. For all $i \neq k$, $S_i^{k-1} = S_i^k$. Thus by the lower property, $|\inf(f(S^{k-1})) - \inf(f(S^k))| \leq |\inf(S_k^{k-1}) - \inf(S_k^k)|$. By the upper property, $|\sup(f(S^{k-1})) - \sup(f(S^k))| \leq |\sup(S_k^{k-1}) - \sup(S_k^k)|$. It follows that $d(f(S^{k-1}), f(S^k)) = |\inf(f(S^{k-1})) - \inf(f(S^k))| + |\sup(f(S^{k-1})) - \sup(f(S^k))| \leq |\inf(S_k^{k-1}) - \inf(S_k^k)| + |\sup(S_k^{k-1}) - \sup(S_k^k)| \leq \varepsilon$.

Because d is a distance, $d(f(S), f(T)) = \sum_{k=1}^n d(f(S^{k-1}), f(S^k)) \leq n\varepsilon = \delta$.

Part Two. I show that strategyproofness implies responsiveness.

Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. For $j \in N$ let $S^j \in \Sigma^N$ such that $S_i^j = S_i$ for $i \leq j$ and such that $S_i^j = T_i$ otherwise. Define $S^0 \equiv S$.

Let $k \in N$. It is sufficient to show that $f(S^{k-1}) \subseteq f(S^k)$.

Because $S_k^{k-1} \subseteq S_k^k$, it follows that $\inf(S_k^k) \leq \inf(S_k^{k-1})$. For all $i \neq k$, $S_i^{k-1} = S_i^k$. Therefore, by the lower property, either $\inf(f(S^k)) = \inf(f(S^{k-1}))$ or else $\inf(S_k^k) \leq \inf(f(S^k)) \leq \inf(f(S^{k-1})) \leq \inf(S_k^{k-1})$. It follows that $\inf(f(S^k)) \leq \inf(f(S^{k-1}))$.

Also because $S_k^{k-1} \subseteq S_k^k$, it follows that $\sup(S_k^{k-1}) \leq \sup(S_k^k)$. Therefore, by the upper property, either $\sup(f(S^k)) = \sup(f(S^{k-1}))$ or else $\sup(S_k^{k-1}) \leq \sup(f(S^{k-1})) \leq \sup(f(S^k)) \leq \sup(S_k^k)$. It follows that $\sup(f(S^{k-1})) \leq \sup(f(S^k))$.

From the fact that $\inf(f(S^k)) \leq \inf(f(S^{k-1})) \leq \sup(f(S^{k-1})) \leq \sup(f(S^k))$ it follows that $f(S^{k-1}) \subseteq f(S^k)$. □

Proof of Proposition 2. Let $f^{p,q}$ be an endpoint rule. Let $S \in \Sigma^N$, $i \in N$, and $\succeq_i \in \mathcal{P}$, and let $S_i^* \in \Sigma$ be the peak of $\succeq_i$. Define $S_{-i} \equiv (S_1, ..., S_i^*, ..., S_n)$. I show that $f(S_{-i}) \succeq_i f(S)$.

First, I show that if $\inf(f(S_{-i})) < \inf(S_i^*)$, then $\inf(f(S)) \leq \inf(f(S_{-i}))$. Let $H^+ \equiv \{j \in N : \inf(S_j) \leq \inf(f(S_{-i}))\}$. By the definition of the endpoint rule and the fact that $\inf(f(S_{-i})) < \inf(S_i^*)$, it follows that $|H^+ \cap (N \setminus i)| \geq p$. Therefore $|H^+| \geq p$ which implies that $\inf(f(S)) \leq \inf(f(S_{-i}))$.

Next, I show that if $\inf(f(S_{-i})) > \inf(S_i^*)$, then $\inf(f(S)) \geq \inf(f(S_{-i}))$. By the definition of the endpoint rule, it follows that it follows that $|G^+(S_{-i}, \inf(f(S_{-i})))| < p$. From the fact that $\inf(f(S_{-i})) > \inf(S_i^*)$ it follows that $i \in G^+(S_{-i}, \inf(f(S_{-i})))$. This implies that $|G^+(S, \inf(f(S_{-i})))| < p$ and therefore that $\inf(f(S)) \geq \inf(f(S_{-i}))$.

Together, this implies that either $\inf(f(S)) \leq \inf(f(S_{-i})) \leq S_i^*$ **or** $\inf(f(S)) \geq \inf(f(S_{-i})) \geq S_i^*$. From a dual argument it follows that either $\sup(f(S)) \leq \sup(f(S_{-i})) \leq S_i^*$ **or** $\sup(f(S)) \geq \sup(f(S_{-i})) \geq S_i^*$. Therefore, $f(S_{-i}) \in \mathcal{B}(S_i^*, f(S))$, and thus $f(S_{-i}) \succeq_i f(S)$. $\square$

Proof of Proposition 3: I will show that strategyproofness implies the independent aggregation of the lower endpoints. That strategyproofness implies the independent aggregation of the upper endpoints follows from a dual argument.

Let $S, T \in \Sigma^N$ such that $\inf(S_i) = \inf(T_i)$ for all $i \in N$ and let f satisfy strategyproofness. I will show that $\inf(f(S)) = \inf(f(T))$. For $j \in N$ let $S^j \in \Sigma^N$ such that $S_i^j = S_i$ for $i \leq j$ and such that $S_i^j = T_i$ otherwise. Define $S^0 \equiv S$.

Let $k \in N$. It is sufficient to prove that $\inf(f(S^{k-1})) = \inf(f(S^k))$.

Because f satisfies strategyproofness it follows from Lemma 2 that f satisfies the lower property. By the lower property, either (a) $\inf(f(S^{k-1})) = \inf(f(S^k))$ or (b) $\inf(f(S^{k-1})) \in B(\inf(S_k^{k-1}), \inf(f(S^k))$ and $\inf(f(S^k)) \in B(\inf(S_k^k), \inf(f(S^{k-1}))$.

That $\inf(f(S^{k-1})) \in B(\inf(S_k^{k-1}), \inf(f(S^k))$ implies that either (i) $\inf(S_k^{k-1}) \geq \inf(f(S^{k-1})) \geq \inf(f(S^k))$ or (ii) $\inf(S_k^{k-1}) \leq \inf(f(S^{k-1})) \leq \inf(f(S^k))$. That $\inf(f(S^k)) \in B(\inf(S_k^k), \inf(f(S^{k-1}))$ implies that either (iii) $\inf(S_k^k) \geq \inf(f(S^k)) \geq \inf(f(S^{k-1}))$ or (iv) $\inf(S_k^k) \leq \inf(f(S^k)) \leq \inf(f(S^{k-1}))$. The combinations of (i) and (iii) and of (ii) and (iv) directly imply that $\inf(f(S^{k-1})) = \inf(f(S^k))$. The combinations of (i) and (iv) and of (ii) and (iii), combined with the fact that $\inf(S_k^{k-1}) = \inf(S_k^k)$, imply that $\inf(f(S^{k-1})) = \inf(f(S^k))$. $\square$

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Appendix for Referees (not for publication)

The "If" part of of Theorem 1. Let $f^{p,q}$ be an endpoint rule. I will show that $f^{p,q}$ satisfies the four axioms.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$, and let $x \in f^{p,q}(S)$. I will show that $x \in f^{p,q}(T)$. Because $x \in f^{p,q}(S)$ it follows that $|G^+(S,x)| \geq p$ and $|G^-(S,x)| \geq q$. Because $S_i \subseteq T_i$ for all $i \in N$ it follows that $G^+(S,x) \subseteq G^+(T,x)$ and $G^-(S,x) \subseteq G^-(T,x)$. This implies that $|G^+(T,x)| \geq p$ and $|G^-(T,x)| \geq q$ and therefore that $x \in f^{p,q}(T)$.

Anonymity: Letπ be a permutation of N, let $S \in \Sigma^N$, and let $x \in f^{p,q}(S)$. Because $x \in f^{p,q}(S)$ it follows that $p \leq |G^+(S,x)| = |G^+(\pi S, x)| \geq p$ and $q \leq |G^-(S,x)| = |G^-(\pi S, x)| \geq q$. Hence $x \in f^{p,q}(\pi S)$. Because the choice of π is arbitrary, it follows that $f^{p,q}(S) = f^{p,q}(\pi S)$ for all $S \in \Sigma^N$.

Continuity: Let $\delta > 0$, let $\varepsilon = \frac{\delta}{3}$, and let $S \in \Sigma^N$. Let $S' \in \Sigma^N$ such that $d(S_i, S'_i) < \varepsilon$ for all $i \in N$. It follows that $|\inf f^{p,q}(S) - \inf f^{p,q}(S')| < \varepsilon$ and $|\sup f^{p,q}(S) - \sup f^{p,q}(S')| < \varepsilon$. As a consequence, $d(f^{p,q}(S), f^{p,q}(S')) < 2\varepsilon < \delta$.

Weak Neutrality: Let $\phi \in \Phi^+$ and $S \in \Sigma^N$. Let $x \in \phi(f(S))$. This implies that $\phi^{-1}(x) \in f(S)$ and therefore that $|G^+(S, \phi^{-1}(x))| \geq p$ and $|G^-(S, \phi^{-1}(x))| \geq q$. Because ϕ is increasing, $G^+(S, \phi^{-1}(x)) = G^+(\phi(S), \phi(\phi^{-1}(x))) = G^+(\phi(S), x)$ and $G^-(S, \phi^{-1}(x)) = G^-(\phi(S), x)$. It follows that $|G^+(\phi(S), x)| \geq p$ and $|G^-(\phi(S), x)| \geq q$ and therefore that $x \in f^{p,q}(\phi S)$. Because the choice of ϕ is arbitrary, it follows that $f^{p,q}(S) = f^{p,q}(\pi S)$ for all $S \in \Sigma^N$. □

The "If" part of of Theorem 2. Let $f^{p,p}$ be a symmetric endpoint rule. By Theorem 1, because $f^{p,p}$ is an endpoint rule it satisfies responsiveness, anonymity, and continuity.

To show that $f^{p,p}$ satisfies strong neutrality, let $\phi \in \Phi$ and $S \in \Sigma^N$. Let $x \in \phi(f(S))$. This implies that $\phi^{-1}(x) \in f(S)$, and therefore that $|G^+(S, \phi^{-1}(x))| \geq p$ and $|G^-(S, \phi^{-1}(x))| \geq p$. If ϕ is increasing, then $G^+(S, \phi^{-1}(x)) = G^+(\phi(S), \phi(\phi^{-1}(x))) = G^+(\phi(S), x)$ and $G^-(S, \phi^{-1}(x)) = G^-(\phi(S), x)$. If ϕ is decreasing, then $G^+(S, \phi^{-1}(x)) = G^-(\phi(S), x)$ and $G^-(S, \phi^{-1}(x)) = G^+(\phi(S), x)$. Hence, because $|G^+(S, \phi^{-1}(x))| \geq p$ and $|G^-(S, \phi^{-1}(x))| \geq p$, it follows that $|G^+(\phi(S), x)| \geq p$ and $|G^-(\phi(S), x)| \geq p$, and therefore that $x \in f^{p,p}(\phi(S))$. Because the choice of ϕ is arbitrary, it follows that $f^{p,p}(S) = f^{p,p}(\phi(S))$ for all $S \in \Sigma^N$. □

The proof of Claim 1 relies on the following definitions. First, let $\mathcal{G}$ be the set of functions $g : \mathbb{R}^N \to \mathbb{R}$. I define the following two axioms on $\mathcal{G}$.

$\mathcal{G}$-anonymity: For every permutation π of N and $x \in \mathbb{R}^N$, $g(x) = g(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

$\mathcal{G}$-strategyproofness: For every agent i with single-peaked preferences $\succeq_i$ over $\mathbb{R}$ and associated peak p_i, and for every $x \in \mathbb{R}^N$, $g(p_i, x_{-i}) \succeq_i g(x)$.

Proof of Claim 1. Let f satisfy anonymity and strategyproofness. This implies, by Proposition 3, that f satisfies independent aggregation of endpoints. Consequently, there exists $\underline{g}, \overline{g} :\in \mathcal{G}$ such that for all $S \in \Sigma^N$, $\inf f(S) = \underline{g}(\{\inf S_i\})$ and $\sup f(S) = \overline{g}(\{\sup S_i\})$.

I show that $\underline{g}$ satisfies $\mathcal{G}$-anonymity and $\mathcal{G}$-strategyproofness. To show that $\underline{g}$ satisfies $\mathcal{G}$-anonymity, let π be a permutation of N and let $x \in \mathbb{R}^N$. Let $S \in \Sigma^N$ such that, for all $i \in N$, $\inf S_i = x_i$. Then $\underline{g}(x) = \inf f(S)$. By anonymity, $f(S) = f(\pi S)$, which implies that $\inf f(S) = \inf f(\pi S)$. By construction $\inf f(\pi S) = \underline{g}(x_{\pi(1)}, \ldots, x_{\pi(n)})$. Consequently, $g(x) = g(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

To show that $\underline{g}$ satisfies $\mathcal{G}$-strategyproofness, let $i \in N$ with single-peaked preferences $\succeq_i$ over $\mathbb{R}$ and with associated peak p_i. Let $x \in \mathbb{R}^N$ and assume by means of contradiction that $g(p_i, x_{-i}) \succeq_i g(x)$ is false. Let $S, T \in \Sigma^N$ such that (a) for all $j \neq i$, $S_j = T_j$ and $\inf S_j = x_j$, (b) $\sup S_i = \sup T_i$, and (c) $\inf S_i = p_i \neq \inf T_i = x_i$. By Lemma 2, because f satisfies strategyproofness it satisfies the lower property, and therefore either (i) $\inf f(S) = \inf f(T)$ or (ii) $\inf(f(S)) \in B(\inf(S_i), \inf(f(T)))$ and $\inf(f(T)) \in B(\inf(T_i), \inf(f(S)))$. If (i) then $g(p_i, x_{-i}) = \inf f(S) = \inf f(T) = g(x)$, which implies that $g(p_i, x_{-i}) \succeq_i g(x)$, If (ii) then $g(p_i, x_{-i}) \in B(p_i, g(x))$, which implies that $g(p_i, x_{-i}) \succeq_i g(x)$. This contradiction establishes that $\underline{g}$ satisfies $\mathcal{G}$-strategyproofness.

Because $\underline{g}$ satisfies $\mathcal{G}$-anonymity and satisfies $\mathcal{G}$-strategyproofness, it follows from Moulin (1980, Proposition 2) that there exist $n+1$ real numbers $\alpha_1, \ldots, \alpha_{n+1} \in \overline{\mathbb{R}}$ such that, for all $x \in \mathbb{R}^N$, $\underline{g}(x) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1})$, where m is the function that selects the median element of $\{x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n+1}\}$.

A parallel argument can be used to establish that $\overline{g}$ satisfies $\mathcal{G}$-anonymity and satisfies $\mathcal{G}$-strategyproofness, and therefore there exist $n+1$ real numbers $\beta_1, \ldots, \beta_{n+1} \in \overline{\mathbb{R}}$ such that, for all $x \in \mathbb{R}^N$, $\overline{g}(x) = m(x_1, \ldots, x_n, \beta_1, \ldots, \beta_{n+1})$. Without loss of generality, we can order the numbers such that $\alpha_1 \leq \ldots \leq \alpha_{n+1}$ and that $\beta_1 \leq \ldots \leq \beta_{n+1}$.

To finish the proof, it is sufficient to show that $\alpha_i < \beta_i$ for all $i \in \{1, \ldots, n+1\}$. Suppose by means of contradiction that this is false. Then there exists $k \leq n+1$ such that $|\{i : \beta_i \leq \alpha_{(k)}\}| \geq k$. Let $c = n + 1 - k$. Let $S \in \Sigma^N$ such that for $i = 1, \ldots, c$, $\inf S_i > \alpha_k$, and for $i = c+1, \ldots, n+1$, $\sup S_i \leq \alpha k$. Then $\inf f(S) = \underline{g}(\{\inf S_i\}) = m(\{\inf S_i\}, \alpha_1, \ldots, \alpha_{n+1}) = \alpha_k \geq m(\{\sup S_i\}, \beta_1, \ldots, \beta_{n+1}) = \sup f(S)$, a contradiction that proves the claim. $\square$

Independence of the Axioms. The independence of the axioms in Theorems 1 and 2 follows from rules 1–4. The independence of the axioms in Corollaries 1 and 2 follows from rules 1–3.

Rule 1: This rule satisfies responsiveness, anonymity, weak neutrality, and strong neutrality, but fails continuity and therefore strategyproofness.

$f(S) = f^{2,2}(S)$ for all S such that $\pi S = \phi S^n$ for some π and ϕ, otherwise $f(S) = f^{1,1}(S)$. (only when $n \geq 3$)

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. If there exists π

and ϕ such that $\pi S = \phi S^n$, then $f(S) = f^{2,2}(S) \subseteq f^{2,2}(T) \subseteq f^{1,1}(T)$. It follows that $f(S) \subseteq f(T)$.

Anonymity: Let $S \in \Sigma^N$ and let π' be a permutation of N. If there exists π and ϕ such that $\pi S = \phi S^n$, then $\pi^*(\pi' S) = \phi S^n$ for $\pi^* = \pi {\pi'}^{-1}$. In this case, $f(\pi' S) = f^{2,2}(\pi' S) = f^{2,2}(S) = f(S)$. If there does not exist π and ϕ such that $\pi S = \phi S^n$, then there does not exist π and ϕ such that $\pi(\pi' S) \neq \phi S^n$. In this case, $f(\pi' S) = f^{1,1}(\pi' S) = f^{1,1}(S) = f(S)$. It follows that for all S, $f(\pi' S) = f(S)$.

Strong neutrality: Let $S \in \Sigma^N$ and let $\phi' \in \Phi$. If there exists π and ϕ such that $\pi S = \phi S^n$, then $\pi(\phi' S) = \phi^* S^n$ for $\phi^* = \phi' \phi$. In this case, $f(\phi' S) = f^{2,2}(\phi' S) = \phi' f^{2,2}(S) = \phi' f(S)$. If there does not exist π and ϕ such that $\pi S = \phi S^n$, then there does not exists π and ϕ such that $\pi(\phi' S) \neq \phi S^n$. In this case, $f(\phi' S) = f^{1,1}(\phi' S) = \phi' f^{1,1}(S) = \phi' f(S)$. It follows that for all S, $f(\phi' S) = f(S)$.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Continuity: Let $n \geq 3$, and let $\varepsilon > 0$ such that $d(S_i, T_i) < 3\varepsilon$ for all $i \in N$ implies that $d(f(S), f(T)) < 0.1$. Let $S, T \in \Sigma^N$ such that $S_1 = T_1 = (0, 2)$, $S_2 = (5 + \varepsilon, 7)$, $T_2 = (5 - \varepsilon, 7)$, and $S_k = T_k = (3, 5)$ and for $k \geq 3$. Then there exists π and ϕ such that $\pi S = \phi S^n$ but there does not exists π and ϕ such that $\pi T = \phi S^n$. It follows that $f(S) = f^{2,2}(S) = (3, 5)$, that $f(T) = f^{1,1}(T) = (0, 7)$, that $d(f(S), f(T)) = 5 > 0.1$, but that $d(S_i, T_i) < 3\varepsilon$ for all $i \in N$, a contradiction.

Rule 2: This rule satisfies anonymity, responsiveness, continuity, and strategyproofness, but fails weak neutrality and strong neutrality.

$f(S) = (1, 2)$ for all $S \in \Sigma^N$.

Anonymity: Let $S \in \Sigma^N$ and let π be a permutation of N. Then $f(S) = (1, 2)$ and $f(\pi S) = (1, 2) = f(S)$.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. Then $f(T) = (1, 2)$ and $f(S) = (1, 2) \subseteq f(T)$.

Continuity. Let $\delta > 0$ and let $\varepsilon > 0$. Then for all S, T such that $d(S_i, T_i) < \varepsilon$ for all $i \in N$, $d(f(S), d(f(T)) = 0 < \delta$.

Weak neutrality. Let $S \in \Sigma^N$ such that $\phi(x) = x + 1$. Then $f(\phi S) = (1, 2)$ but $\phi f(S) = (2, 3)$, a contradiction.

Strong neutrality. Any rule that fails weak neutrality necessarily fails strong neutrality.

Strategyproofness. The outcome is variant to changes in the intervals and is therefore strategyproof.

Rule 3: This rule satisfies responsiveness, continuity, weak neutrality, strong neutrality, and strategyproofness, but fails anonymity.

$f(S) = S_1$ for all $S \in \Sigma^N$.

Responsiveness: Let $S, T \in \Sigma^N$ such that $S_i \subseteq T_i$ for all $i \in N$. Then $S_1 \subseteq T_1$ which implies that $f(S) \subseteq f(T)$.

Continuity. Let $\delta > 0$ and let $\varepsilon = \delta$. Then for all S, T such that $d(S_i, T_i) < \varepsilon$ for all $i \in N$, $d(f(S), f(T)) = d(S_1, T_1) < \delta$.

Strong neutrality: Let $S \in \Sigma^N$ and let $\phi \in \Phi$. Then $f(\phi S) = \phi S_1$ and $\phi f(S) = \phi S_1$, which implies that $f(\phi S) = \phi f(S)$.

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Anonymity: Let $S \in \Sigma^N$ such that $S_1 \neq S_2$ and let π be a permutation such that $\pi(1) = 2$. Then $f(S) = S_1$ but $f(\pi S) = S_2 \neq S_1$.

Strategyproofness. Agent 1 will reveal her preferred interval, and the rule is invariant to the other agents.

Rule 4: This rule satisfies anonymity, continuity, weak neutrality, and strong neutrality, but fails responsiveness and therefore strategyproofness.

$f(S) = (\inf f^{1,n}(S), \max\{\sup f^{1,n}(S), \inf S_1, \ldots, \inf S_n\})$ for all $S \in \Sigma^N$.

Anonymity: Let $S \in \Sigma^N$ and let π be a permutation of N. Then $f(\pi S) = (\inf f^{1,n}(\pi S), \max\{\sup f^{1,n}(\pi S), \inf \pi S_1, \ldots, \inf \pi S_n\})$. Note that $\inf f^{1,n}(\pi S) = \inf f^{1,n}(S)$, $\sup f^{1,n}(\pi S) = \sup f^{1,n}(S)$, and $\{\inf \pi S_1, \ldots, \inf \pi S_n\} = \{\inf S_1, \ldots, \inf S_n\}$, which implies that $f(\pi S) = f(S)$.

Continuity. Let $\delta > 0$ and let $\varepsilon = \frac{\delta}{3}$. Then for all S, T such that $d(S_i, T_i) < \varepsilon$ for all $i \in N$, $|\inf f^{1,n}(S) - \inf f^{1,n}(T)| < \varepsilon$ and $|\max\{\sup f^{1,n}(S), \inf S_1, \ldots \inf S_n\} - \max\{\sup f^{1,n}(T), \inf T_1, \ldots \inf T_n\}| < 2\varepsilon$, which implies that $d(f(S), f(T)) < 3\varepsilon = \delta$.

Strong neutrality. Let $S \in \Sigma^N$ and let $\phi \in \Phi$.

Then $f(\phi S) = (\inf f^{1,n}(\phi S), \max\{\sup f^{1,n}(\phi S), \inf \phi S_1, \ldots, \inf \phi S_n\}) =$
$(\phi \inf f^{1,n}(S), \max\{\phi \sup f^{1,n}(S), \phi \inf S_1, \ldots, \phi \inf S_n\}) =$
$\phi(\inf f^{1,n}(S), \max\{\sup f^{1,n}(S), \inf S_1, \ldots, \inf S_n\}) = \phi f(S)$,

Weak neutrality: Any rule that satisfies strong neutrality necessarily satisfies weak neutrality.

Responsiveness: Let $S \in \Sigma^N$ such that $S_i = (0, 2i)$ for all $i \in N$. Then $S_i^n \subseteq S_i$ for all $i \in N$. However, $f(S^n) = (0, 2i - 1)$ while $f(S) = (0, 2)$, contradicting the fact that $f(S^n) \subseteq f(S)$. $\square$