Inconsistency on Multimember Courts

Alan D. Miller* and Shiran Rachmilevitch‡

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Abstract

Appellate courts sometimes issue inconsistent decisions. Individual judges are sometimes inconsistent too. We argue that making judges more consistent could exacerbate the problem of inconsistent courts. We do so through a variant of Arrow's model of preference aggregation in which preferences are complete, but need not be transitive. We introduce an ordinal rationality measure to compare preference relations. Using this measure, we introduce a new axiom, monotonicity in rationality, which requires the collective preference to become more rational when the individual preferences become more rational. We show that no collective choice rule satisfies monotonicity in rationality and the standard Arrovian assumptions: unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

JEL Codes: D60; D70; D71; K40.

1 Introduction

Appellate courts sometimes issue inconsistent decisions. One reason, of course, is that judges are inconsistent. It should not be surprising that a court with inconsistent judges will, at times, yield inconsistent decisions. This suggests a path forward: find a way to make individual judges more consistent, and it must follow, as the night the day, that courts will become more consistent.

*Faculty of Law, Western University, 1151 Richmond Street, London, Ontario N6A 3K7, Canada. Email: alan.miller@uwo.ca Web: http://alandmiller.com

†Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: shiranrach@econ.haifa.ac.il Web: https://sites.google.com/site/profshiranrachmilevitch/

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The purpose of this paper is to demonstrate that this logic, while appealing, is incorrect. It is not true that the consistency of multimember courts follows from that of its judges. This is not true regardless of whether the court uses majority rule to make collective decisions. We prove that there is no reasonable procedure that multimember courts can use to ensure that more consistent judges will not lead to less consistent verdicts.

We will make these arguments precise later in the paper. First, however, we need to explain the sense in which we use the term "consistency." As others have noted (Easterbrook, 1982; Kornhauser and Sager, 1986), there are many possible meanings of this term. We identify judicial consistency with the economists' notion of rationality of a preference relation.[1] It requires that sets of judgments must be consistent in relation to each other, as opposed to requiring decisions to be internally consistent. As noted by Easterbrook (1982), rationality (specifically transitivity) is a necessary condition for consistency; that is, a failure of rationality implies inconsistency. While it is not a sufficient condition—a decision can be inconsistent on its own, and not only in relation to other decisions—rationality is largely separable from these other forms of consistency.

We note that it is possible, at least in theory, to affect the consistency of individual judges. The cost (or difficulty) of doing so depends on the reason for the inconsistency For example, judges may be inconsistent for reasons akin to behavioral explanations of failures of rationality: judges fail to be consistent because limitations on cognition lead them to rely on heuristics that result in biased decision making.[2] It may be possible to reduce these inconsistencies at low cost by informing the judge of the inconsistency and letting them correct their behavior, or through "nudges." Some such inconsistencies, however, may be more costly to address. If it is possible to affect the consistency of individual judges, it is important to understand the implications of doing so.

An alternate explanation stems from the theory of law as a multi-criterial choice process, pioneered by Spitzer (1979). If finding the correct legal decision requires the balancing of several hard-to-compare values, then inconsistency of judicial decisions is an implication of the Arrow Impossibility Theorem (Arrow, 1963). Inconsistencies that arise from the problem of multi-criterial choice cannot be remedied except through more fundamental changes to the nature of law.[3]

[1]We do not take a stance on whether this relation reflects a "preference" or a "judgment" in the sense of Kornhauser and Sager (1986). By "rationality" we refer to the extent to which a preference relation satisfies transitivity or other "coherence" conditions such as quasi-transitivity, acyclicity, semi-transitivity, and the interval order. By "rational" we refer to a preference relation that satisfies transitivity. For more on coherence, see Bossert and Suzumura (2007).

[2]As a note, the behavioral literature suggests that the cognitive biases will themselves be consistent, while the resulting judgments will not be.

[3]See also Katz and Sandroni (2017). In the related context of loopholes, Katz (2010) argues that inconsistencies arising from the nature of multi-criterial choice should not be viewed as normatively undesirable. It is harder to make a similar argument for inconsistencies that arise from cognitive

Our starting point is an observation of Easterbrook (1982) that multimember courts may behave in an inconsistent manner even if all judges are individually consistent.[4] Easterbrook's argument follows from the Arrow Impossibility Theorem: it is generally impossible to aggregate a set of rational preference relations in a reasonable way, where "reasonable" requires satisfying several well defined axioms. Arrow's argument, however, says nothing about degrees of rationality, and does not tell us what happens when individual preferences become more (or less) rational.

In the Arrovian model of preference aggregation (Arrow, 1963), individual preferences are assumed to be reflexive, complete, and transitive. We begin by modifying Arrow's model to remove the requirement that individual preference relations be transitive or satisfy other known coherence conditions.[5] Having removed this strict rationality requirement, we can formulate our problem as a question about collective choice: do more rational individuals create a more rational society?[6]

To talk about "more" and "less" rational (or "more" and "less" coherent), we need the concept of a rationality measure introduced by Afriat (1973). A rationality measure is a means by which preference relations can be compared in terms of their coherence. Rather than pick a specific such measure, our argument will apply to any one of a broad class of such measures.

So suppose that we have a set of agents, at least one of whom is less than fully rational, with a not-necessarily-rational collective preference. We may be able to induce an agent to "correct" his preferences. When the agent becomes more rational as a result of this correction, will the collective preference become more rational as a result?

Our conclusion is negative; if group decisions are made in a non-dictatorial way, it is possible that an increase in individual rationality may lead to a decrease in collective rationality. It may be possible to manipulate a court (or a jury) by helping individual judges (or jurors) correct their mistakes. Outside of the legal context, a group of people may become more susceptible to "Dutch books" when the individuals' susceptibility lessens.

We illustrate this problem by means of a simple example. It is clear that many collective choice rules satisfy the remaining assumptions imposed by Arrow (1963):

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biases of judges.

[4]This is not the only reason that inconsistencies may arise on multimember courts. Another is the "Doctrinal Paradox" of Kornhauser and Sager (1986), which relates to problems of internal consistency of opinions of multimember courts.

[5]With a similar motivation, de Clippel (2014) studies mechanism design without the assumption that individuals make choices as if they are maximizing a preference relation.

[6]Following Easterbrook (1982), we identify the court—the "society"—with a *preference-aggregation method* that satisfies the classic Arrovian restrictions. We recognize, of course, that this general question may be studied in a different setting, and that even those following the aggregation approach may address the question without the imposition of Arrow's conditions. We do not attempt to provide the "ultimate answer" to this question. Rather, we provide one meaningful answer within one of the most basic frameworks in economics.

unrestricted domain, weak Pareto, independence of irrelevant alternatives, and non-dictatorship. A simple example is the method of majority decision, in which alternative x is weakly preferred to alternative y whenever the majority weakly prefers x to y. (For more on the method of majority decision, see Sen (1964, 1966).)

However, the method of majority decision has an undesirable property. Suppose that there are three individuals, Alice, who prefers x to y, y to z, and x to z, Bob, who prefers z to x, x to y, and z to y, and Carol who prefers x to z, z to y, and y to x. Alice and Bob have transitive preferences, but Carol does not. By the method of majority decision, x is preferred to y, z is preferred to y, and x is preferred to z, leading to a transitive and rational collective preference. However, suppose that Carol realizes that her preferences are irrational and seeks to "correct" them. She decides to retain her view that y is preferred to x but changes her opinion of z, so that she now prefers y to z and z to x. As a consequence, the method of majority decision leads to the collective preference x to y, y to z, and z to x, and is no longer transitive. In this case, the collective preference became less rational *because* Carol became more rational.

We show that this problem is not unique to method of majority decision. In fact, every collective choice rule which satisfies the remaining assumptions of Arrow (1963) will have this undesirable property. For every such method it is possible that an increase in the rationality of individual preferences will lead to a decrease in the rationality of the collective preference.[7]

Our formal model can be described as follows. First, we study a modified version of collective choice rules in which neither the individual nor collective preferences are required to be rational.[8] We assume only that preference relations be reflexive and complete. Thus each individual's preferences can be described by a reflexive and complete relation, and the collective preference can be described by a reflexive and complete relation as well. We implicitly assume that every possible combination of individual preference relations is possible; *i.e.* that Arrow's unrestricted domain axiom holds in this setting.

Using accepted notions of rationality, we introduce the concept of an ordinal rationality measure, and identify some minimal conditions that any reasonable rationality measure should satisfy. Our conditions are satisfied by all known measures of rationality such as the measures of Afriat (1973), Houtman and Maks (1985), and Varian (1990), as well as the money pump index (Echenique et al., 2011) and the minimal swaps index (Apesteguia and Ballester, 2015). We formulate an axiom, *monotonicity in rationality*, to address the problem exhibited by majority rule in the above exam-

[7]For this reason, we do not argue that the method of majority decision is any worse than any other method in this context. Several studies, including Sen (1966), Inada (1969), and Batra and Pattanaik (1972), examine the conditions under which pairwise majority does not lead to cycles. Dasgupta and Maskin (2008) provide an argument that the method of majority decision is more robust than other voting methods in that it violates the standard axioms on fewer domains.

[8]While we study the aggregation of preference relations, all of our results would hold if the paper was formulated with respect to choice functions.

ple. This axiom requires that the collective choice rule be monotonic with respect to the rationality measure; that is, if individual preferences change and become more rational, then the collective preference should become more rational, if it changes at all.

In addition to monotonicity in rationality, we impose the three additional axioms of Arrow (1963): *weak Pareto*, which requires that society strictly prefer x to y whenever every individual strictly prefers x to y, *independence of irrelevant alternatives*, which requires that a change in the opinions about alternative z does not affect the relative ranking of alternatives x and y, and *nondictatorship*, which requires that no individual be a dictator. We show that the four axioms are incompatible. In other words, regardless of which ordinal rationality measure we choose, we cannot find a collective choice rule which is monotonic, weakly Paretian, independent of irrelevant alternatives, and nondictatorial.

1.1 Related literature in social choice:

Previous studies have sought to weaken the assumption of rationality in Arrow (1963) by permitting a wider range of collective preferences. The case of quasi-transitive collective preferences was studied by many including Gibbard (1969), Sen (1969, 1970), Schick (1969), and Mas-Colell and Sonnenschein (1972), and that of acyclic preferences by Mas-Colell and Sonnenschein (1972), Blau and Deb (1977), Deb (1981), and Blair and Pollak (1982). For more see Sen (1977).[9]

Other scholars have tried to avoid the negative conclusions of Arrow (1963) by moving in the opposite direction. Instead of expanding the range of admissible collective preferences, these studies restrict the domain of allowable preferences. The most prominent example is that of the single-peaked preference restriction of Black (1948a,b) and Arrow (1963).

The most closely related formal literature is the study of tournaments, which are described by binary relations which are antisymmetric and complete. Unlike the preference relations we study, tournaments do now allow for the possibility of ties. In this context, Monjardet (1978) shows that a collective choice rule that (a) maps every profile of transitive preferences into a transitive preference, (b) satisfies the independence of irrelevant alternatives axiom and (c) satisfies a non-imposition axiom is either dictatorial or "persecutive." Roughly speaking, persecutive means that the decisive coalitions are all the coalitions that do not contain a certain individual i. A related result can be found in Barthelemy (1982). As far as we can tell, the monotonicity in rationality axiom that we present is new to this paper.

[9] A related literature looks at the case in which preferences are transitive but not necessarily complete (see Baucells and Shapley, 2008; Pini et al., 2009).

2 Model and result

Let X be a set of alternatives, $|X| \geq 3$. A preference relation R on X is (a) **complete** if for all $x, y \in X$, $x \neq y$ implies that either xRy or yRx, (b) **reflexive** if for all $x \in X$, xRx, and (c) **transitive** if for all $x, y, z \in X$, xRy and yRz implies that xRz. Let $\mathscr{R}$ be the set of all complete and reflexive preference relations on X. A **preference ordering** is a preference relation which is complete, reflexive, and transitive. Let $\mathscr{R}^* \subseteq \mathscr{R}$ be the set of preference orderings on X.

For a preference relation $R \in \mathscr{R}$ we denote by P its asymmetric component; that is, xPy if xRy but not yRx. A preference relation is **acyclic** if, for every $k \geq 3$ and every $x^1, ..., x^k \in X$, $x^i P x^{i+1}$ for all $i < k$ implies that $x^1 R x^k$.[10] Let $\mathscr{R}^a \subseteq \mathscr{R}$ be the set of preference relations which are complete, reflexive, and acyclic. It is well known that $\mathscr{R}^* \subsetneq \mathscr{R}^a \subsetneq \mathscr{R}$ (see Suzumura, 1983).

For $Y \subseteq X$, denote by $\mathscr{R}|_Y$ the set all complete and reflexive preference relations on Y, and denote by $\mathscr{R}^*|_Y$ the set all preference orderings in $\mathscr{R}|_Y$. For $R \in \mathscr{R}$ and $Y \subseteq X$, denote by $R|_Y \in \mathscr{R}|_Y$ the restriction of R to Y.

Let $N \equiv \{1, ..., n\}$ be a finite set of agents, $n \geq 2$. A *profile* $\mathbf{R} = (R_1, \cdots, R_n) \in \mathscr{R}^N$ is a vector of preference relations, one for each agent. A **collective choice rule** is a mapping $f \colon \mathscr{R}^N \to \mathscr{R}$.[11] We define $R_0 \equiv f(\mathbf{R})$ to be the social relation, and we denote by P_0 its asymmetric component. A set of elements $Y \subseteq X$ is **top-ranked** in profile $\mathbf{R}$ if $a \in Y$, $b \in X \setminus Y$, and $i \in N$ implies that aP_ib.

A **rationality measure**[12] is a binary relation $\succcurlyeq$ on $\mathscr{R}$ which satisfies the following properties:

1. For all $R \in \mathscr{R}$, $R \succcurlyeq R$.

2. For all $R' \in \mathscr{R}^*$ and $R \in \mathscr{R}$, $R \succcurlyeq R'$ implies that $R \in \mathscr{R}^*$.

3. For all $R^* \in \mathscr{R}^*$ and $R \in \mathscr{R} \backslash \mathscr{R}^a$: if there is a three-element set $Y \subseteq X$ which is top-ranked in both relations, such that $R^* \big|_{X \setminus Y} = R \big|_{X \setminus Y}$, then $R^* \succcurlyeq R$.

For two profiles $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$ we write $\mathbf{R} \succcurlyeq \mathbf{R}'$ if $R_i \succcurlyeq R'_i$ for all $i \in N$.

Property 1, known as reflexivity, requires each preference relation to be "at least as rational" as itself. Property 2 requires that only a transitive preference ordering can be at least as rational as another transitive preference ordering. Property 3 requires that every transitive preference ordering must be at least as rational as every cyclic

[10] For ease of exposition, we have chosen a definition that assumes completeness.

[11] This is a slight change from the standard definition, in which the domain of a collective choice rule is a set of preference orderings (see Sen, 1970).

[12] This definition is broad and only contains minimal conditions for a rationality measure. Arguably, a measure should also be transitive. The breadth of the definition is desirable because it implies a weaker monotonicity in rationality axiom, below. The definition is not used for any other purpose in this paper.

preference relation, provided that the two relations are identical except for the three top-ranked elements.

A wide range of rationality measures satisfies these conditions. We provide two examples. The simplest rationality measure $\succcurlyeq'$ is one for which $R^* \succcurlyeq' R$ if and only if (a) $R^* \in \mathscr{R}^*$ and $R \in \mathscr{R} \setminus \mathscr{R}^*$ or (b) $R^* = R$. A more complicated rationality measure can incorporate the structure of coherence properties studied in the social choice literature. For example, we can define a rationality measure $\succcurlyeq''$ such that $R^* \succcurlyeq'' R$ if and only if (a) there exists an $\mathscr{C} \in \{\mathscr{R}^*, \mathscr{R}^a\}$ such that $R^* \in \mathscr{C}$ but $R \notin \mathscr{C}$ or (b) $R^* = R$.[13]

Our first axiom, monotonicity in rationality, requires that if preference relations change, and each individual's new preference relation stays at least as rational as it was before the change, then the social preference must stay at least as rational.

Monotonicity in rationality: For all $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$, if $\mathbf{R} \succcurlyeq \mathbf{R}'$ then $R_0 \succcurlyeq R_0'$.

The following three axioms were introduced by Arrow (1963); for brevity, we will not discuss them.

Weak Pareto: For every $\mathbf{R} \in \mathscr{R}^N$ and $x, y \in X$, if xP_iy for all $i \in N$, then xP_0y.

Independence of Irrelevant Alternatives: For all $Y \subseteq X$ and $\mathbf{R}, \mathbf{R}' \in \mathscr{R}^N$, if $\mathbf{R}|_Y = \mathbf{R}'|_Y$, then $R_0|_Y = R_0'|_Y$

An individual $d \in N$ is a *dictator* if, for all $\mathbf{R} \in \mathscr{R}^N$, xP_dy implies that xP_0y.

Non-Dictatorship: There does not exist a dictator.

We can now turn to the main result. The proof is given in the appendix.

Theorem 1. *There does not exist a collective choice rule that satisfies monotonicity in rationality, weak Pareto, independence of irrelevant alternatives, and non-dictatorship. Furthermore, the axioms are independent.*

3 Conclusion

We argue an increase in the consistency of individual judges can lead to a decrease in the consistency of courts. We do so by identifying consistency with transitivity of a preference relation, and departing from the standard approach to preference aggregation in three ways. First, we modify the standard model of preference aggregation to

[13]The three classes were chosen for the ease of the exposition. Clearly a rationality measure can incorporate any number of classes, and these not be totally ordered through set inclusion. In particular, the rationality measure can incorporate the coherence properties of quasi-transitivity, semi-transitivity and the interval order. See Cato (2012).

study the case in which neither individual nor collective preferences are required to satisfy transitivity or other coherence conditions. Second, we introduce the concept of an ordinal rationality measure which can be used to compare preference relations in terms of their level of coherence. Third, using this measure, we introduce a monotonicity in rationality axiom that requires the collective preference to become more rational when the individual preferences become more rational. We show that for any ordinal rationality measure, it is impossible to find a collective choice rule which satisfies the monotonicity in rationality axiom and the other standard assumptions introduced by Arrow (1963): unrestricted domain, weak Pareto, independence of irrelevant alternatives, and nondictatorship.

One might argue that inconsistencies in court decisions are not a problem if these inconsistencies arise from the multi-member nature of appellate courts. As long as the public can see that the individual judges are behaving in a consistent manner, inconsistent judgments may be thought of as a form of bad luck, rather than as unfair. However, there are still several problems that remain. First, judges will still be susceptible to problems akin to agenda manipulation (See Levine and Plott, 1977; Plott and Levine, 1978), where one might try to make a judge more consistent in order to get a less consistent result. Second, the source of the inconsistency is not always apparent; judges do not always write concurring and dissenting opinions, stating their reasons in full, and on some courts, judges are not allowed to write concurring or dissenting opinions at all. Third, a similar aggregation problem exists with jurors, whose decision process is not at all visible to the outside world.

On a more technical level, a natural question involves the extent to which the monotonicity in rationality axiom introduced in this paper substitutes for the standard assumption of transitivity. For example, consider an axiom, "transitive-to-transitive" which requires that every profile of transitive preference relations must map to a transitive social relation.[14] There is no logical relation between this axiom and the monotonicity in rationality axiom we propose. For example, a constant rule that maps all profiles to the same non-transitive social preference satisfies monotonicity in rationality but not this axiom. To see that a rule may satisfy the transitive-to-transitive axiom but not monotonicity in rationality, consider a rule in which the social preference coincides with that of the first agent when that agent, and only that agent, has a non-transitive preference relation, and which otherwise maps to a fixed transitive social preference. When all agents' preferences are transitive, this rule will lead to a transitive social preference, and thus it satisfies the transitive-to-transitive axiom. If the first agent's preferences change and become non-transitive, the social preference will clearly become non-transitive. However, if a second agent's preferences change and become non-transitive, the social preference will change back to the original transitive preference, thus violating monotonicity in rationality.

However, in the presence of weak Pareto, independence of irrelevant alternatives, and non-dictatorship, the two axioms are equivalent. To see this, note that in the con-

[14]We thank Eric Maskin for suggesting this idea.

text of Arrow (1963), the transitive-to-transitive axiom implies that Arrow's condition of unrestricted domain is satisfied on the set of transitive profiles. Consequently, when combined with weak Pareto and independence of irrelevant alternatives, this axiom implies the existence of an individual d who is decisive over every pair of alternatives for every transitive profile. That is xP_dy implies xP_0y for every transitive profile. By the independence of irrelevant alternatives axiom, however, it becomes irrelevant whether the profile is transitive; and hence individual d is a dictator.

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Appendix

To prove Theorem 1 we make use of the following lemma. For a coalition $K \subseteq N$, we define $x\bar{D}_K y$ as the statement that the coalition K is decisive for x over y; that is, if xP_iy for all $i \in K$, then xP_0y. Similarly, we define xD_Ky as the statement that the coalition K is decisive for x over y when all others are opposed; that is, if xP_iy for all $i \in K$ and yP_ix for all $i \notin K$, then xP_0y.

Lemma 1. *If a collective choice rule f satisfies monotonicity in rationality, weak Pareto, and independence of irrelevant alternatives, then whenever xD_Ky for a coalition $K \subseteq N$ and some pair of alternatives $x, y \in X$, it follows that $w\bar{D}_Kz$ for every pair $w, z \in X$.*

Proof of Lemma 1. Let the collective choice rule f satisfy the monotonicity in rationality, weak Pareto, and independence of irrelevant alternatives axioms. Let $K \subseteq N$ and $x, y \in X$ such that xD_Ky.

Step one. We claim that, for all $z \in X \setminus \{x, y\}$, if $\mathbf{R} \in \mathscr{R}^N$ such that (a) $R_i\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$ for all $i \in N$, (b) xP_iy for all $i \in K$, and (c) $R_i\big|_{\{x,y,z\}} = R_j\big|_{\{x,y,z\}}$ for all $i, j \in K$, then $R_0\big|_{\{x,y,z\}} \in \mathscr{R}^*\big|_{\{x,y,z\}}$.

To prove this claim, let $z \in X \setminus \{x, y\}$ and let $\mathbf{R} \in \mathscr{R}^N$ satisfying (a), (b), and (c). From the independence of irrelevant alternatives axiom we can assume, without loss of generality, that the set $\{x, y, z\}$ is top-ranked in each R_i and that $R \in \mathscr{R}^{*N}$. Let $\mathbf{R}^\circ \in \mathscr{R}^N$ such that (i) $R_i^\circ = R_i$ for all $i \in K$, (ii) $yP_i^\circ x$, $xP_i^\circ z$, and $zP_i^\circ y$ for all $i \notin K$, and (iii) $\mathbf{R} \succcurlyeq \mathbf{R}^\circ$. Because $xD_K y$ it follows that $xP_0^\circ y$.

From condition (c) it follows that there are two cases: either $xP_i^\circ z$ for all $i \in K$, or $zR_i^\circ x$ for all $i \in K$. In the former case, $xP_i^\circ z$ for all $i \in N$, which implies (by weak Pareto), that $xP_0^\circ z$. Because $xP_0^\circ y$ and $xP_0^\circ z$, it follows that $R_0^\circ \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$. In the latter case, $zP_i^\circ y$ for all $i \in N$, which implies (by weak Pareto), that $zP_0^\circ y$. Because $xP_0^\circ y$ and $zP_0^\circ y$, it follows that $R_0^\circ \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$. Because $R_0^\circ \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$ it follows from monotonicity in rationality and independence of irrelevant alternatives that $R_0 \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$, proving the claim.

Step two. Let $\mathbf{R}' \in \mathscr{R}^{*N}$ such that, for all $i \in K$, $xP_i' y$ and $yP_i' z$ and, for all $i \notin K$, $yP_i' x$ and $yP_i' z$. Because $xD_k y$ it follows that $xP_0' y$, and because $yP_i' z$ for all $i \in N$ it follows from weak Pareto that $yP_0' z$. Because $\mathbf{R}'$ satisfies requirements (a), (b), and (c) of step one, it follows that $R_0' \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$ and therefore $xP_0' z$. By the independence of irrelevant alternatives axiom, this implies that $x\bar{D}_K z$. In other words:

$$xD_K y \text{ implies that } x\bar{D}_K z. \tag{1}$$

Now, let $\mathbf{R}'' \in \mathscr{R}^{*N}$ such that, for all $i \in K$, $zP_i'' x$ and $xP_i'' y$ and, for all $i \notin K$, $zP_i'' x$ and $yP_i'' x$. Because $xD_k y$ it follows that $xP_0'' y$, and because $zP_i'' x$ for all $i \in N$ it follows from weak Pareto that $zP_0'' x$. Because $\mathbf{R}''$ satisfies requirements (a), (b), and (c) of step one, it follows that $R_0'' \big|_{\{x,y,z\}} \in \mathscr{R}^* \big|_{\{x,y,z\}}$ and therefore $zP_0'' y$. By the independence of irrelevant alternatives axiom, this implies that $z\bar{D}_K y$. In other words:

$$xD_K y \text{ implies that } z\bar{D}_K y. \tag{2}$$

By interchanging y and z in statement (2) it follows that $xD_K z$ implies that $y\bar{D}_K z$, and by replacing x by y, y by z, and z by x in statement (1) it follows $yD_K z$ implies that $y\bar{D}_K x$. As a consequence, it follows that

$$xD_K y \text{ implies that } y\bar{D}_K x. \tag{3}$$

By interchanging x and y in statements (1), (2), and (3), it follows that $yD_K x$ implies that $y\bar{D}_K z$, $z\bar{D}_K x$, and $x\bar{D}_K y$. As a consequence, we are led to the implication that for every $\{x, y, z\} \subseteq X$, if $xD_K y$ then $a\bar{D}_K b$ for every $a, b \in \{x, y, z\}$. If $|X| = 3$ this completes the proof.

If $|X| \geq 4$, then let $w \in X \setminus \{x, y, z\}$. By replacing y with z and z with w in statement (2), it follows that $xD_K z$ implies that $w\bar{D}_K z$, concluding the proof.[15] □

Proof of Theorem 1. Let f be a collective choice rule that satisfies the monotonicity in rationality, weak Pareto, independence of irrelevant alternatives, and non-dictatorship

[15]We thank an anonymous referee for simplifying the proof.

axioms. We will derive a contradiction.

Let $S \subseteq N$ be a decisive coalition of minimal size, so that $|T| < |S|$ implies that xD_Ty is false for all $x, y \in X$. By the weak Pareto axiom, such a coalition S exists. By the non-dictatorship axiom and Lemma 1, $|S| \geq 2$. Without loss of generality, let xD_Sy. Let $S_1 \subseteq S$ such that $|S_1| = 1$, let $S_2 \equiv S \setminus S_1$, and let $S_3 \equiv N \setminus S$.

Let $\mathbf{R} \in \mathscr{R}^{*N}$ be a transitive profile such that (a) xP_iy, yP_iz, and xP_iz for all $i \in S_1$, (b) zP_ix, xP_iy, and zP_iy for all $i \in S_2$, and (c) yP_iz, zP_ix, and yP_ix for all $i \in S_3$.

Let $R_\times \in \mathscr{R}$ such that $xP_\times y$, $yP_\times z$, and $zP_\times x$. Let $R_+ \in \mathscr{R}$ such that xP_+z, zP_+y, and yP_+x.

Let $\mathbf{R}^A, \mathbf{R}^B, \mathbf{R}^C \in \mathscr{R}^N$ be profiles such that (a) $R_i^A = R_i^B = R_i^C = R_\times$ for all $i \in S_1$, (b) $R_i^A = R_i^B = R_i^C = R_+$ for all $i \in S_2$, and (c) $R_i^A = R_\times$, $R_i^B = R_+$, and $R_i^C = R_i$ for all $i \in S_3$.

Because of the independence of irrelevant alternatives axiom, we can assume, without loss of generality, that the elements $x, y, z \in X$ are top-ranked in profiles $\mathbf{R}$, $\mathbf{R}^A$, $\mathbf{R}^B$, and $\mathbf{R}^C$ and that $\mathbf{R}\big|_{X\setminus\{x,y,z\}} = \mathbf{R}^A\big|_{X\setminus\{x,y,z\}} = \mathbf{R}^B\big|_{X\setminus\{x,y,z\}} = \mathbf{R}^C\big|_{X\setminus\{x,y,z\}}$. It follows that $\mathbf{R} \succcurlyeq \mathbf{R}^A$, $\mathbf{R} \succcurlyeq \mathbf{R}^B$, and $\mathbf{R} \succcurlyeq \mathbf{R}^C$. Suppose, contrariwise, that R_0 is not transitive. It follows from monotonicity in rationality that neither R_0^A, R_0^B, nor R_0^C may be transitive. Because S_2 is not a decisive coalition, it follows that xR_0^Ay, yR_0^Az, and zR_0^Ax. Because R_0^A is not transitive it follows that $S_1 \cup S_3$ must be decisive for at least one of the three pairs x over y, y over z, or z over x. By Lemma 1, it follows that $xD_{S_1\cup S_3}y$ for all $x, y \in X$.

Because S_1 is not a decisive coalition, it follows that xR_0^Bz, zR_0^By, and yR_0^Bx. Because R_0^B is not transitive it follows that $S_2 \cup S_3$ must be decisive for at least one of the three pairs x over z, z over y, or y over x. By Lemma 1, it follows that $xD_{S_2\cup S_3}y$ for all $x, y \in X$.

Because $xD_{S_1\cup S_3}y$ for all $x, y \in X$ it follows that yP_0^Cz and zP_0^Cx. Because $xD_{S_2\cup S_3}y$ for all $x, y \in X$ it follows that yP_0^Cx. Therefore, R_0^C is transitive, which is a contradiction that proves that R_0 must be transitive.

By assumption, the coalition $S = S_1 \cup S_2$ is decisive for x over y. This implies that xP_0y. Because zP_iy only for $i \in S_2$ and S_2 is not decisive, it follows that yR_0z. Because R_0 is transitive, it follows that xP_0z. But this means that $xD_{S_1}z$, which implies, by Lemma 1, that S_1 is a dictator. This violates the non-dictatorship axiom, and concludes the impossibility proof.

Independence of the Axioms. We describe four collective choice rules. Each of the rules satisfies three of the axioms while violating the fourth. This is sufficient to prove the independence of the axioms.

Rule 1. For all $x, y \in X$, let xR_0y if and only if $|\{i \in N : xR_iy\}| \geq |\{i \in N : yR_ix\}|$. This rule clearly satisfies weak Pareto, independence of irrelevant alternatives, and non dictatorship, but violates monotonicity in rationality.

Rule 2. Let $d \in N$. For all $x, y \in X$, let xR_0y if and only if xR_dy. This rule clearly satisfies monotonicity in rationality, weak Pareto, and independence of irrelevant alternatives, but violates non-dictatorship.

Rule 3. Let $\mathscr{R}^T$ be the set of preference relations such that $R \in \mathscr{R}^T$ and $R' \succcurlyeq R$ implies that $R' \in \mathscr{R}^T$. If $R_1, R_2 \in \mathscr{R}^T$, let $f(R_1, ..., R_n) = R_1$, otherwise, let $f(R_1, ..., R_n) = R_2$. This rule satisfies monotonicity in rationality, weak Pareto, and non dictatorship, but violates independence of irrelevant alternatives.

Rule 4. For all $x, y \in X$, let xR_0y. This rule clearly satisfies monotonicity in rationality, independence of irrelevant alternatives, and nondictatorship, but violates weak Pareto. $\square$